deep imaging

Lecture 3: From continuous to discrete functions

Machine Learning and Imaging

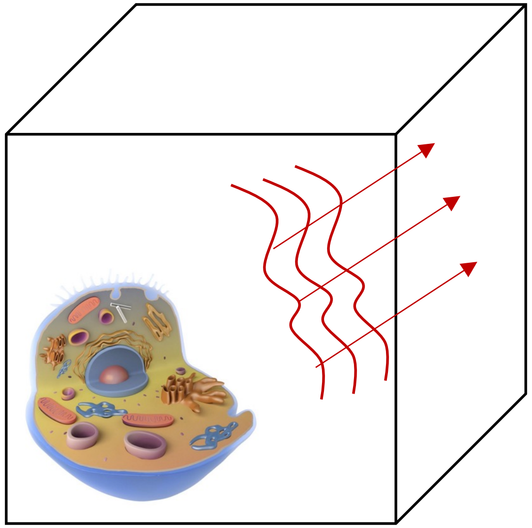
BME 548L

Roarke Horstmeyer

- Fourier transforms
- Convolution theorem
- Sampling theorem

ML+Imaging pipeline introduction

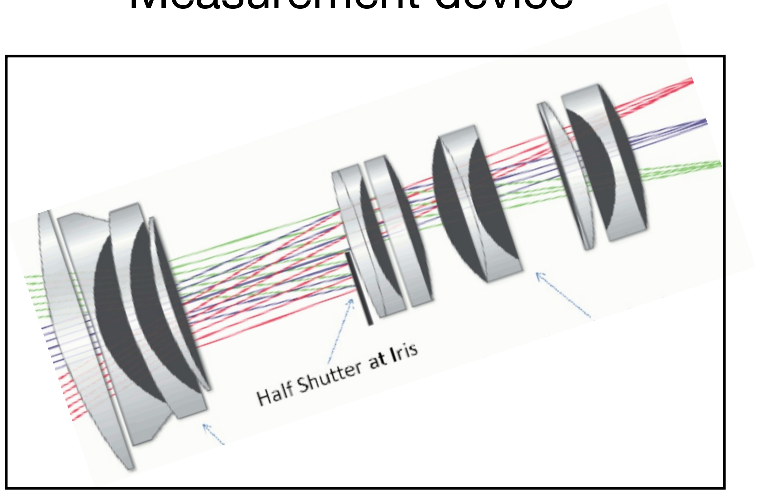
Real World



Continuous
complex fields

(last class) →

Measurement device



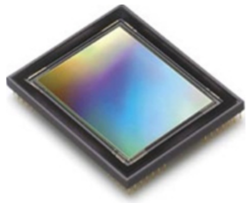
Black box transformations

- Convolution
- Fourier Transform

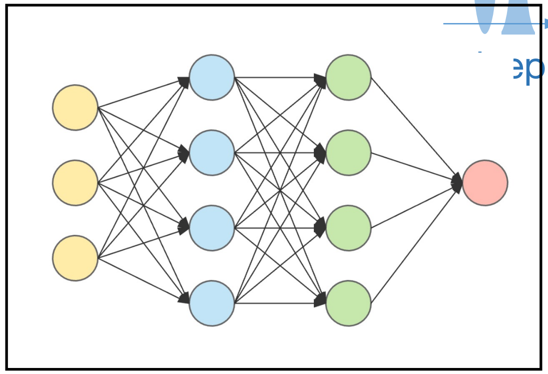
(last class, this class)

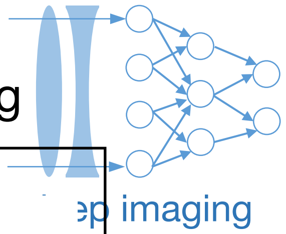
Digitization

$\gamma \rightarrow e^{-}$



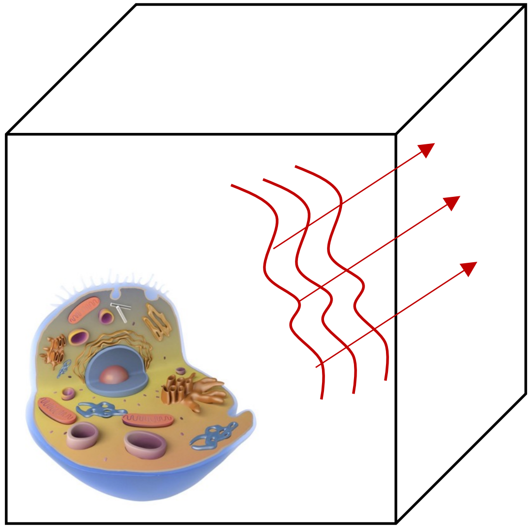
Machine Learning





ML+Imaging pipeline introduction

Real World



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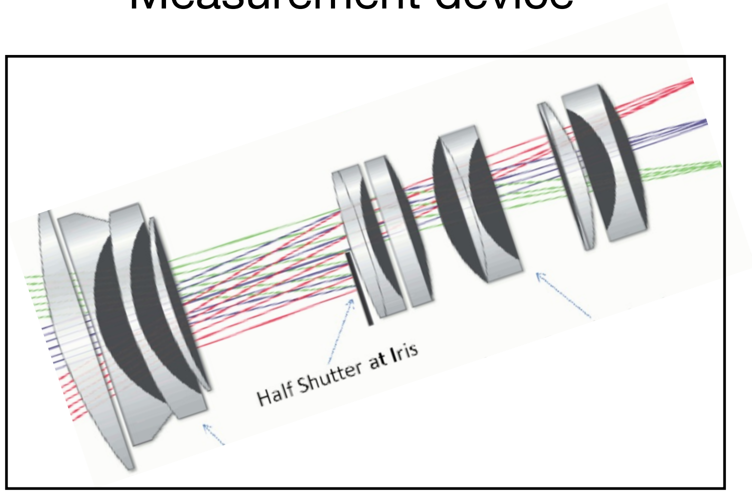

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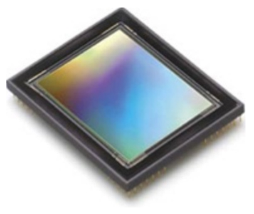
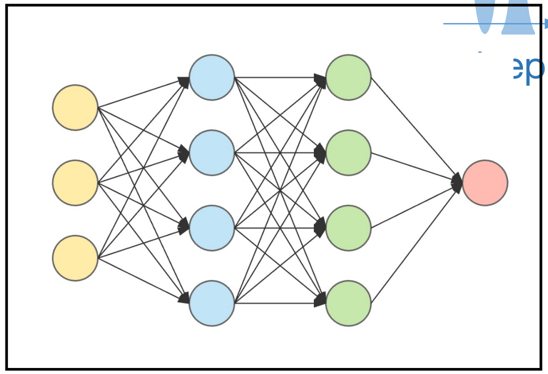


Sampling Theorem

Discrete math &
Linear algebra

(next class)

Machine Learning



ML+Imaging pipeline introduction

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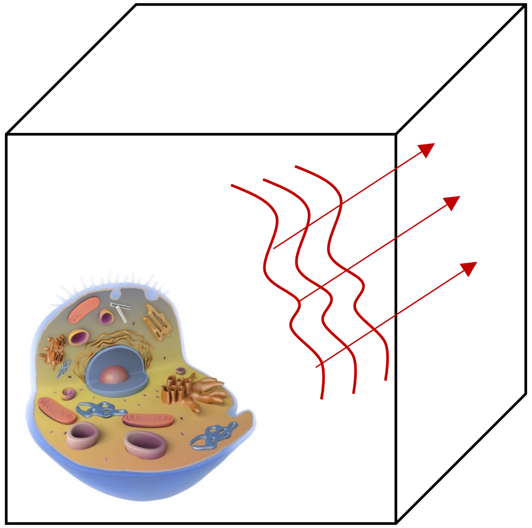

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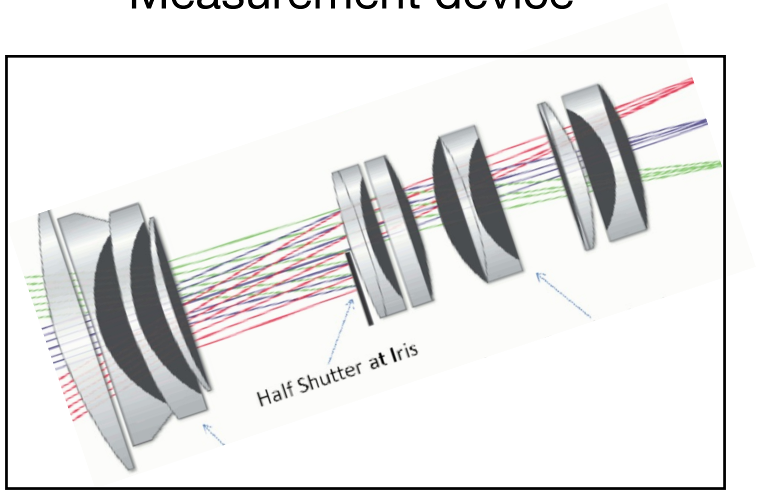

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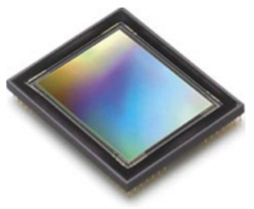
γ -> e⁻



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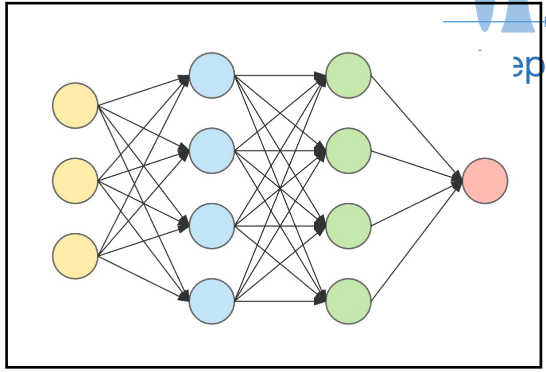

Optimization

Linear classification

Logistic classifier

Neural networks

Convolutional NN's

(next few weeks)

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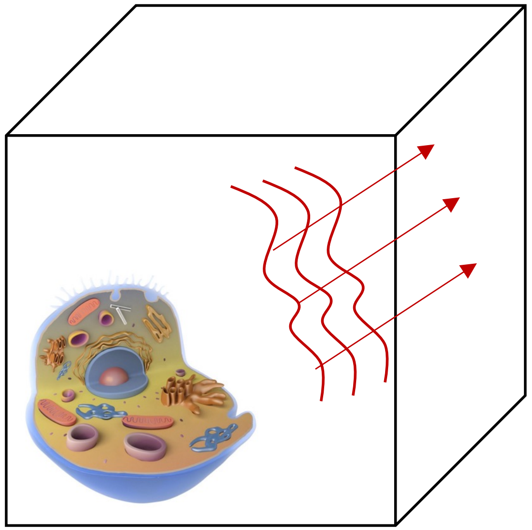

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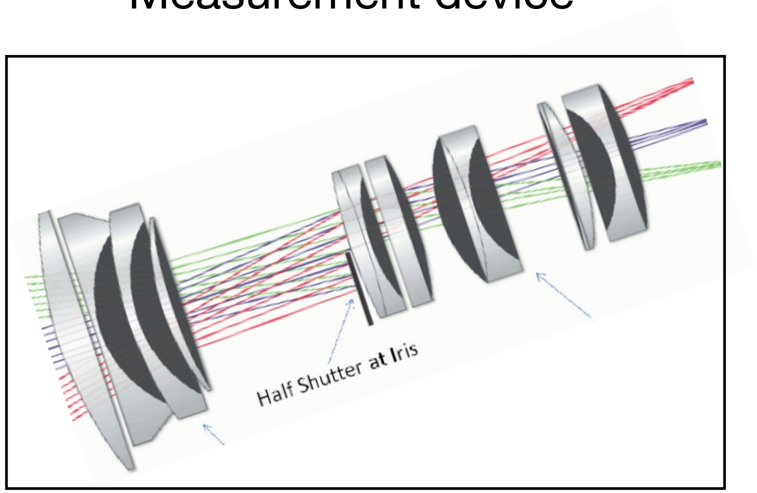

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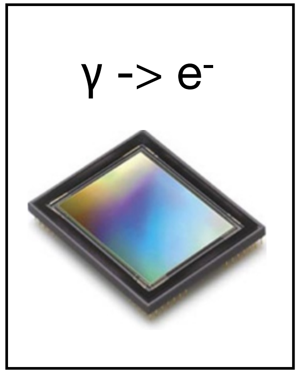

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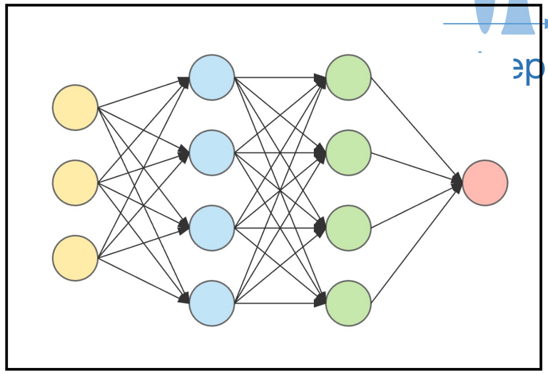

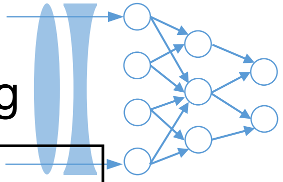

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Month 2

Month 3

Signals in space and spatial frequency

- What we have so far:
  - Continuous & (possibly) complex function for images across space
  - Black-box linear transformation from one domain to the next via convolution

Signals in space and spatial frequency

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Complex function of time -> frequency

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deep imaging

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Fourier Transforms

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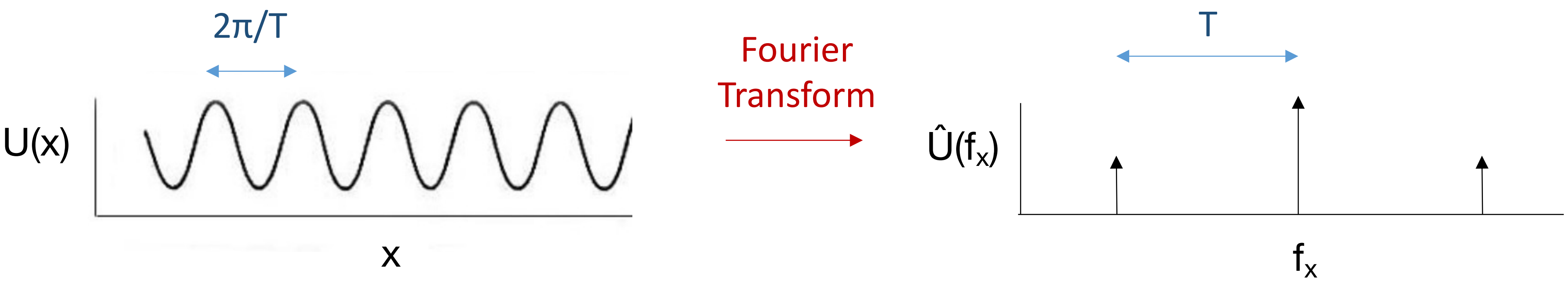

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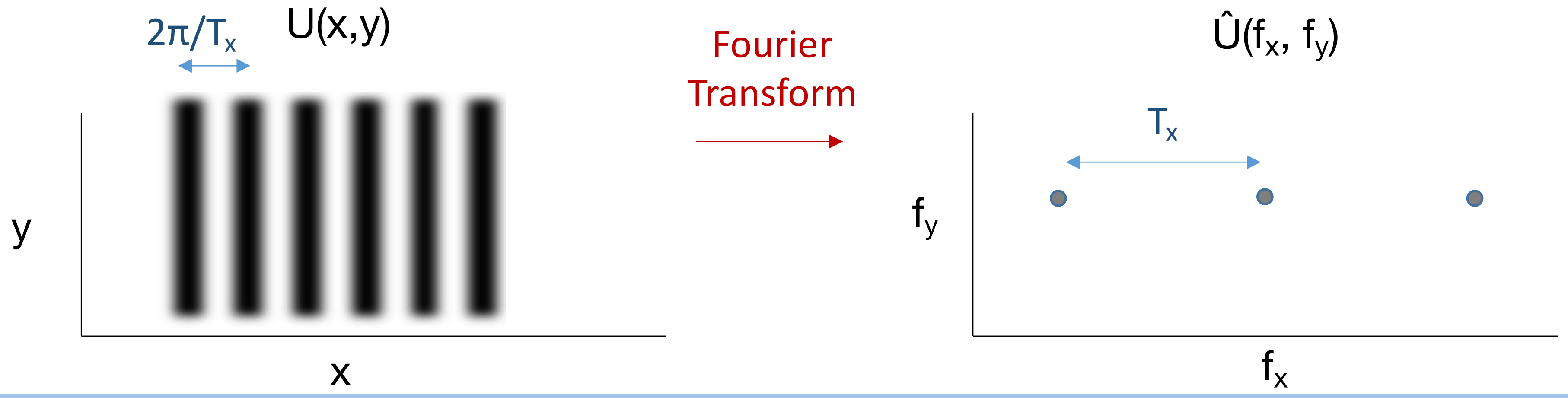

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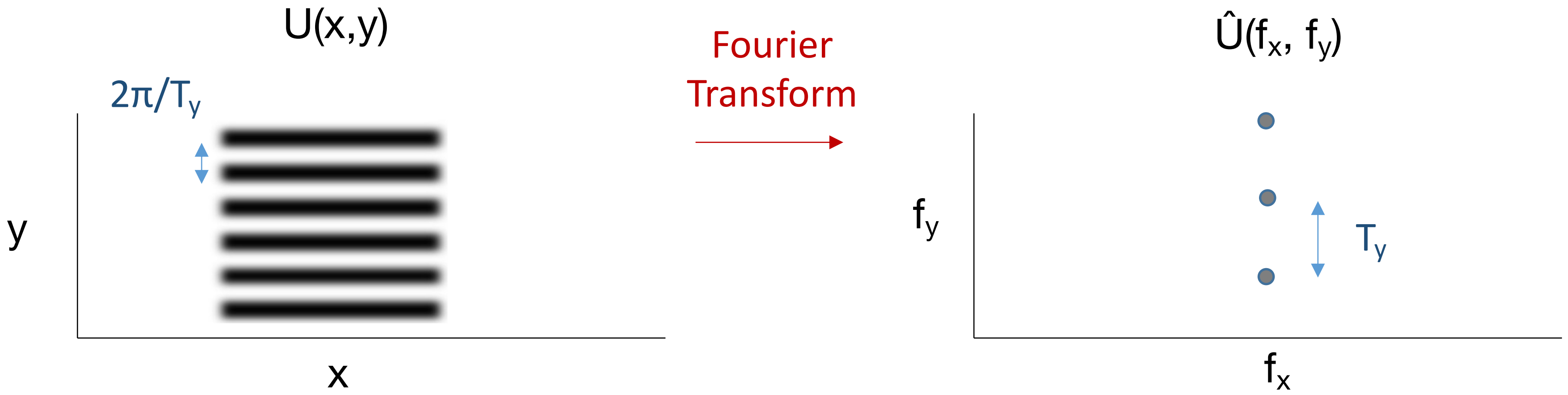

Continuous Fourier transforms – for 2D images

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Decomposition of a signal into elementary functions of form, $\exp\left(-2\pi i(f_x x + f_y y)\right)$:

$$\mathcal{F}\{U(x,y)\} = \hat{U}(f_x, f_y) = \iint_{-\infty}^{\infty} U(x,y) \exp\left(-2\pi i(f_x x + f_y y)\right) dx\, dy$$

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U is absolutely integrable & no infinite discontinuities. The inverse Fourier transform is,

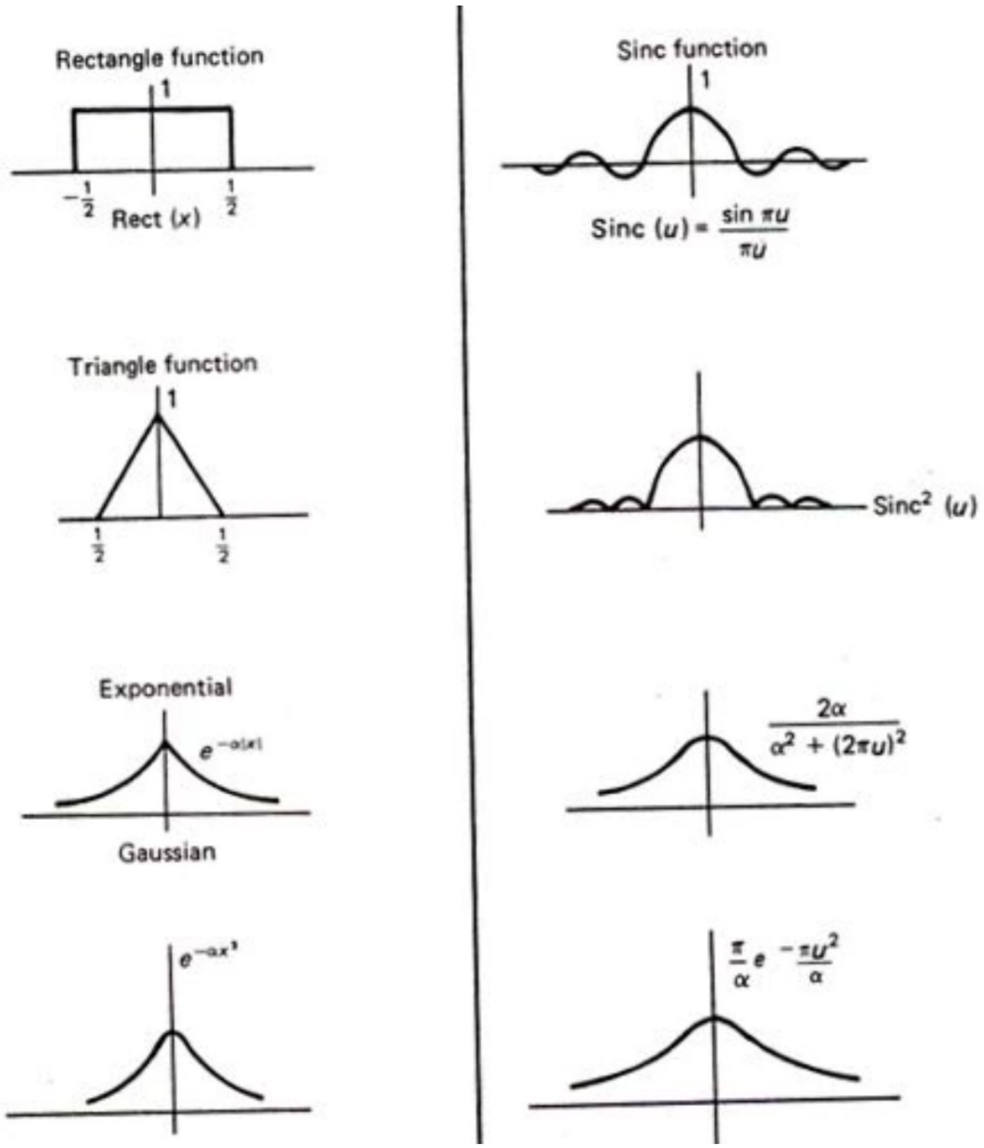
$$\mathcal{F}^{-1}\{\hat{U}(f_x, f_y)\} = U(x,y) = \iint_{-\infty}^{\infty} \hat{U}(f_x, f_y) \exp\left(2\pi i(f_x x + f_y y)\right) df_x\, df_y$$

Additional Details:

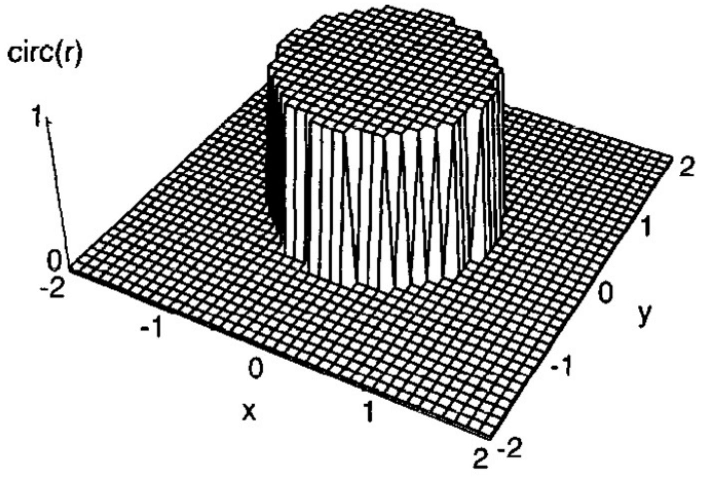
- Goodman Chapter 2.1
- Mathworld/Wikipedia, Fourier Transform

A few examples of Fourier transform pairs, 1D

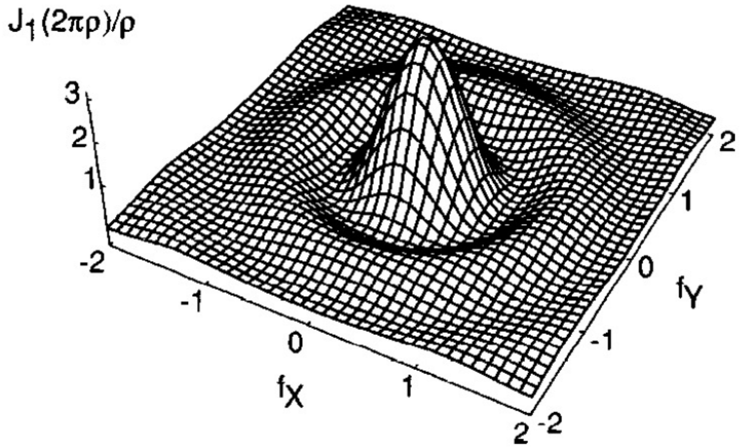
$U(x)$ $\hat{U}(f_x)$



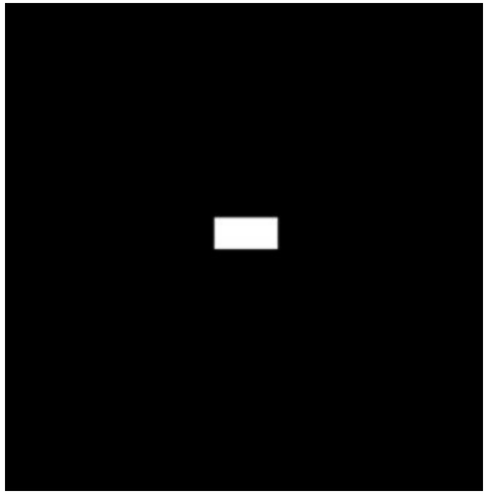
Examples of Fourier transform pairs, 2D

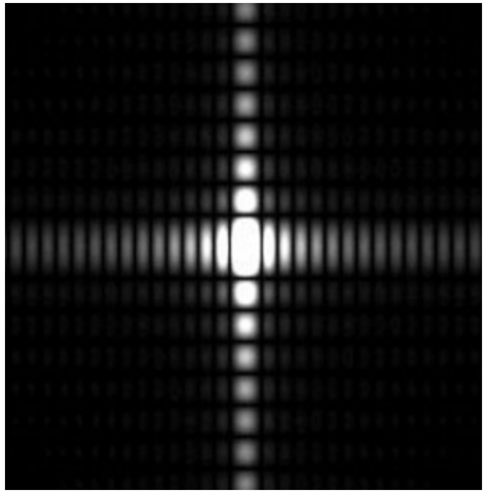


(a)



(b)



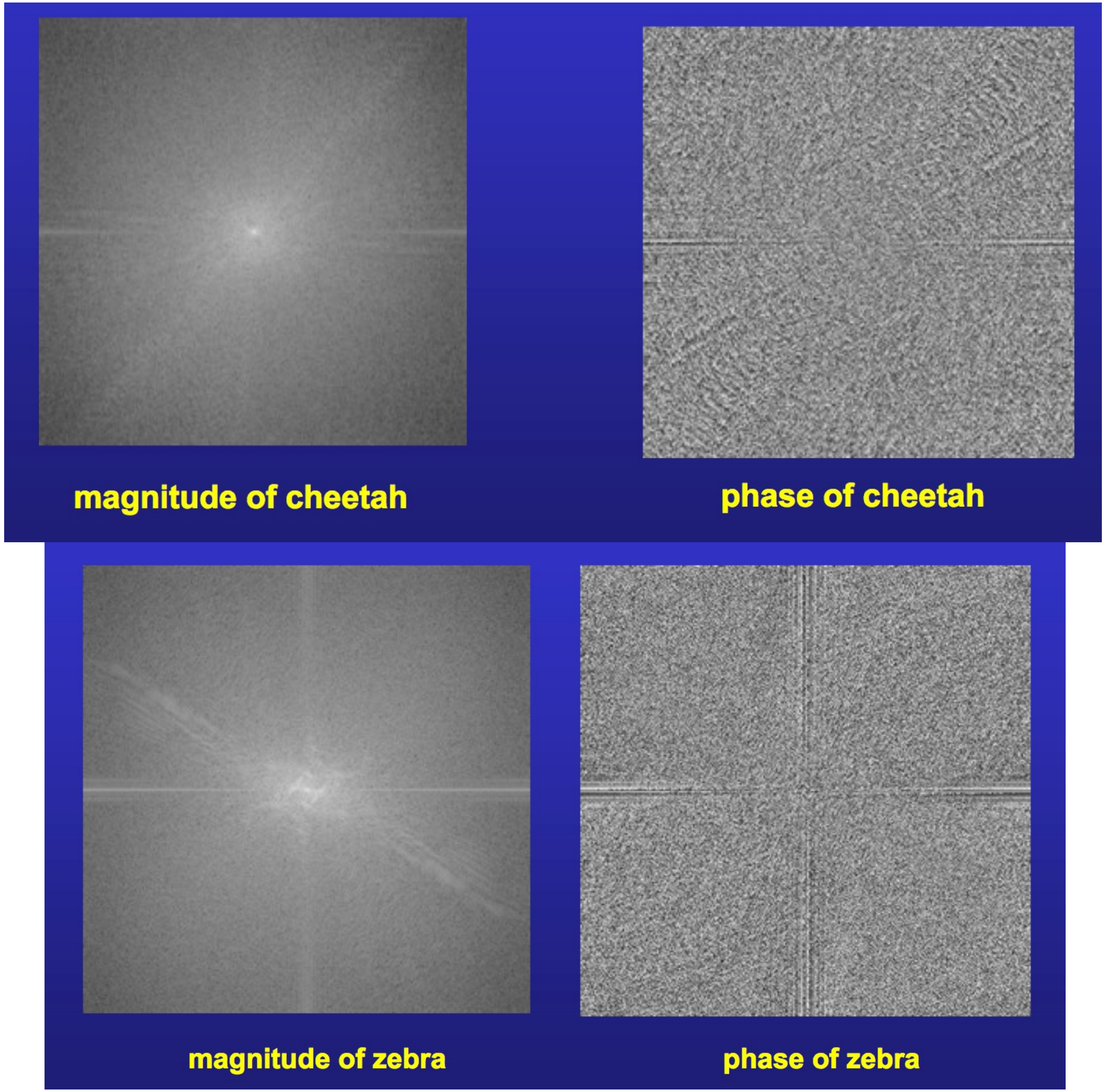


$U_1(x,y)$ $U_2(x,y)$

Cheetah

Zebra

$\hat{U}_1(f_x, f_y)$



$\hat{U}_2(f_x, f_y)$

Important properties of the Fourier transform

- Linearity $h(x) = af(x) + bg(x) \rightarrow H(f) = aF(f) + bG(f)$
- Scaling $h(x) = f(ax) \rightarrow H(f) = (1/a)\, F(f/a)$

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- Scaling $h(x) = f(ax) \rightarrow H(f) = (1/a)\, F(f/a)$
- Shift $h(x) = f(x-a) \rightarrow H(f) = F(f)\, \exp(-2\pi iaf)$
- Parseval's Theorem (energy conservation) $\int |h(x)|^2\, dx = \int |H(f)|^2\, df$
- Fourier inversion theorem

Additional Details:

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Convolution - Fourier Transform relationship: Convolution Theorem

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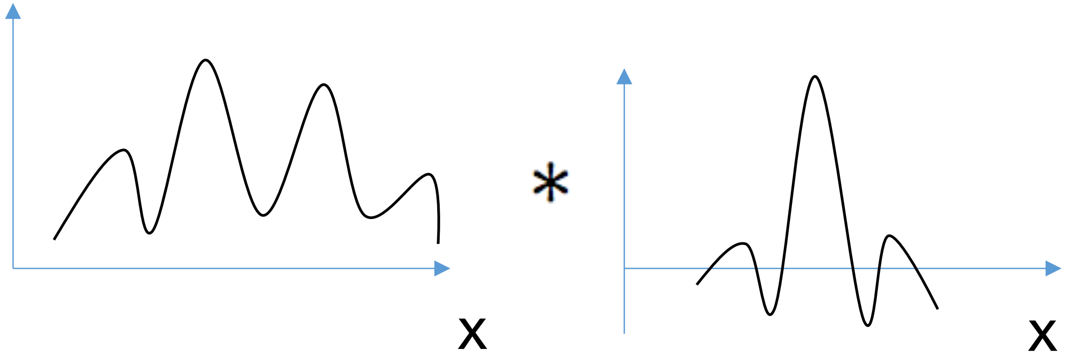
Convolution theorem. If $\mathcal{F}\{g(x, y)) = G(f_X, f_Y)$ and $\mathcal{F}\{h(x, y)\} = H(f_X, f_Y)$, then

$$\mathcal{F}\left\{\iint_{-\infty}^{\infty} g(\xi, \eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta\right\} = G(f_X, f_Y)\, H(f_X, f_Y).$$

"The convolution of two functions in space can be performed by a multiplication in the Fourier domain (spatial frequency domain)"

Example of convolution theorem, 1D

$U_1(x)$ $\quad * \quad$ $U_2(x)=\mathrm{sinc}(ax)$



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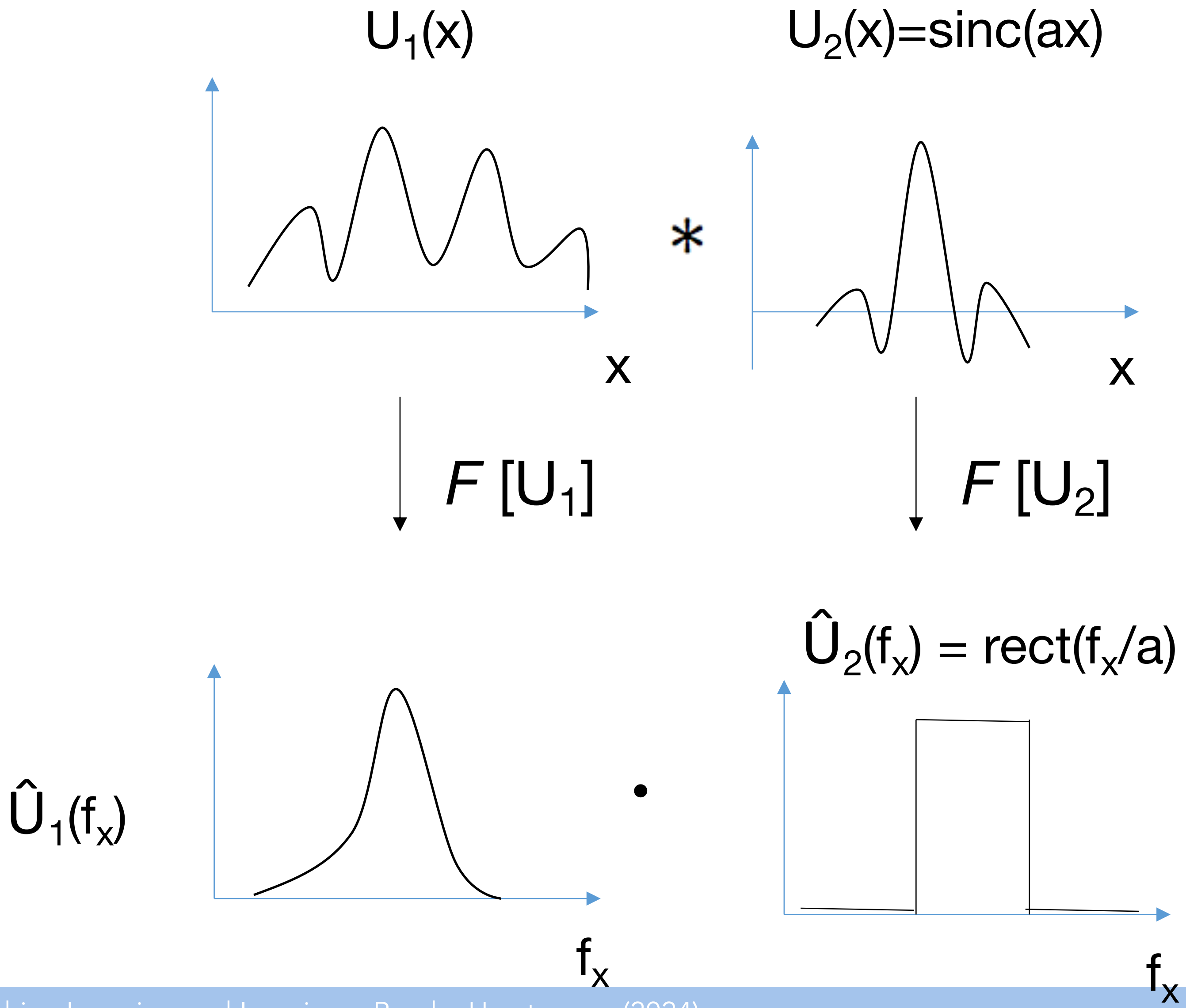

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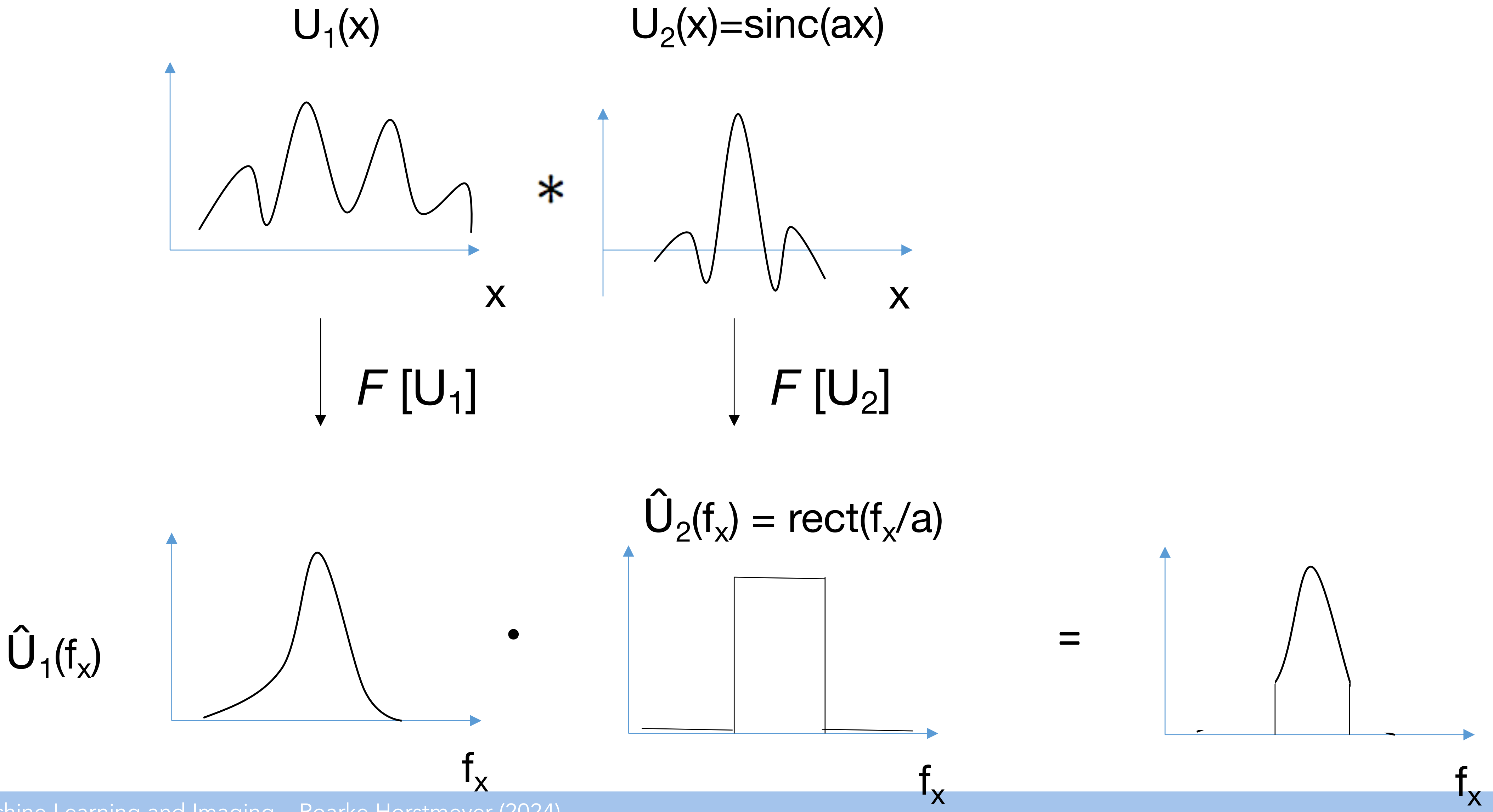

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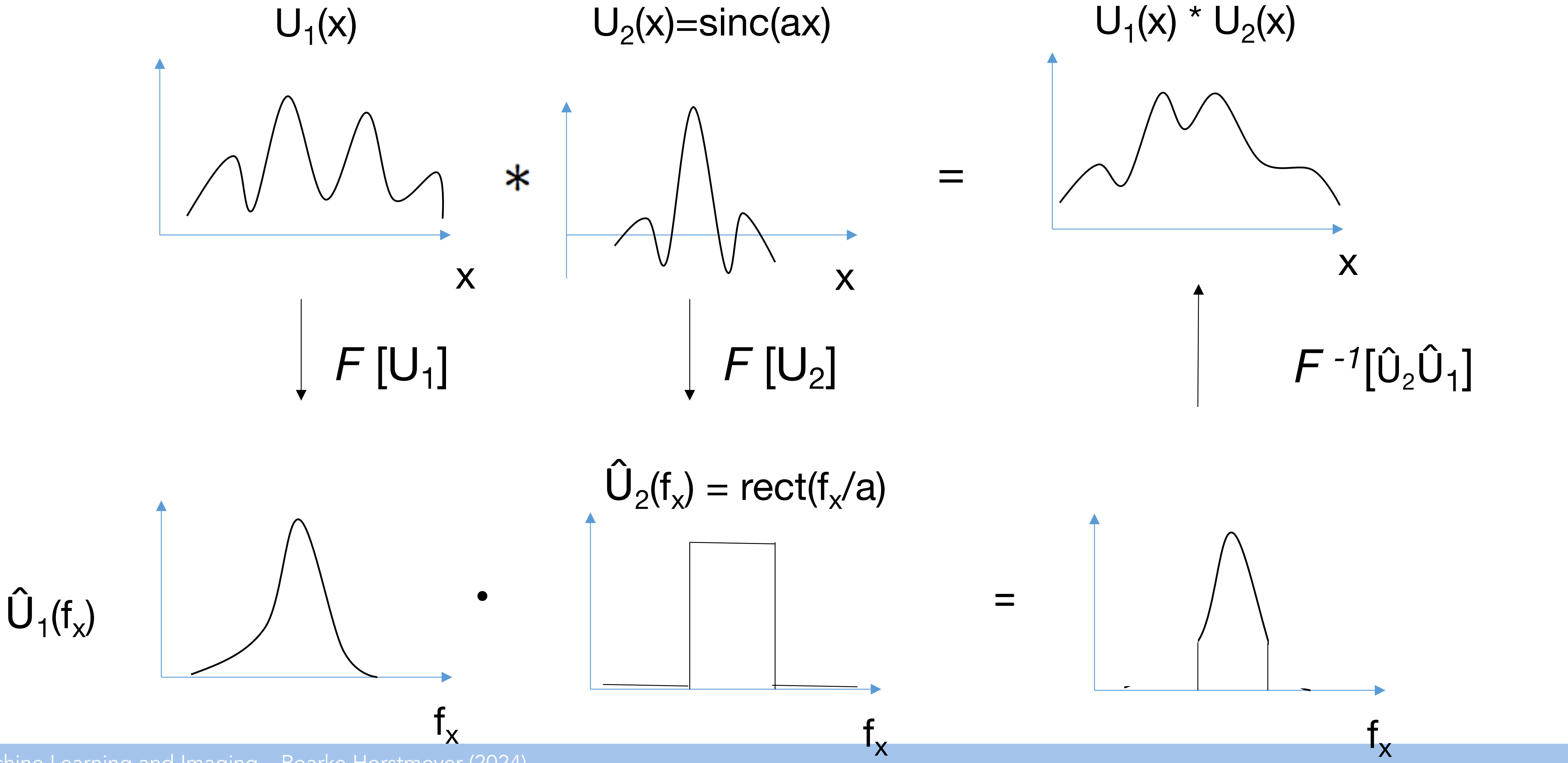
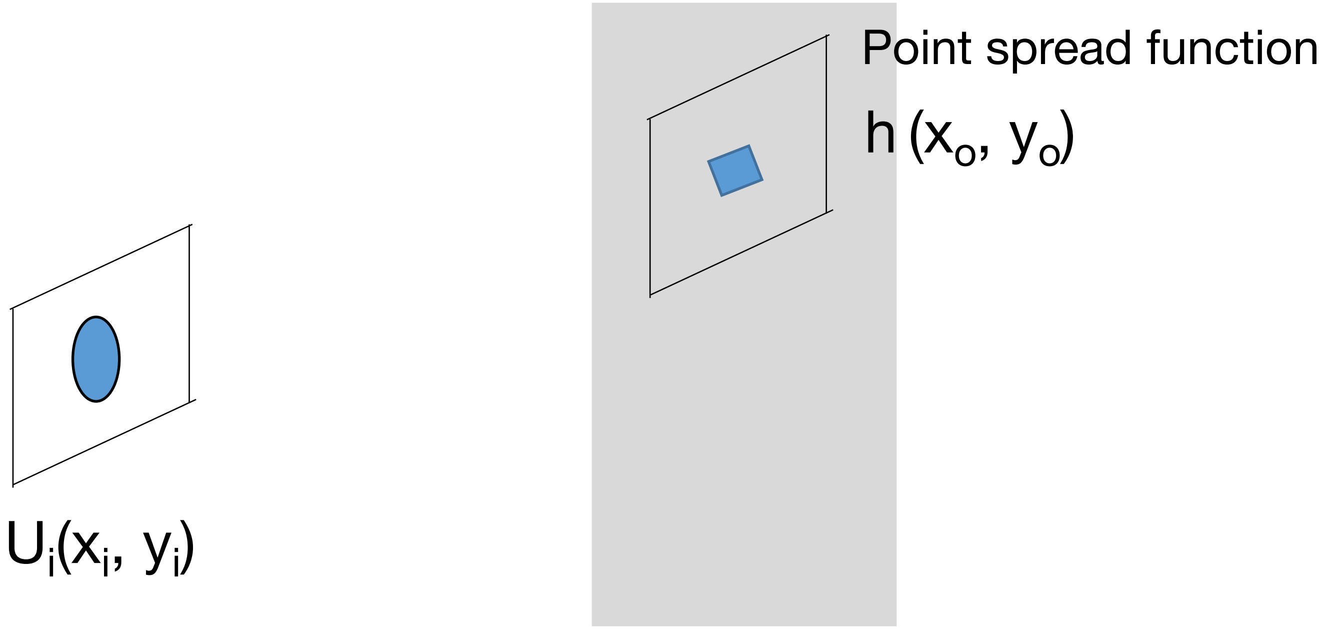


Black box transforms as convolution OR multiplication of frequencies

Knowing the point-spread function, it is direct to model any output of the black box, given an input:



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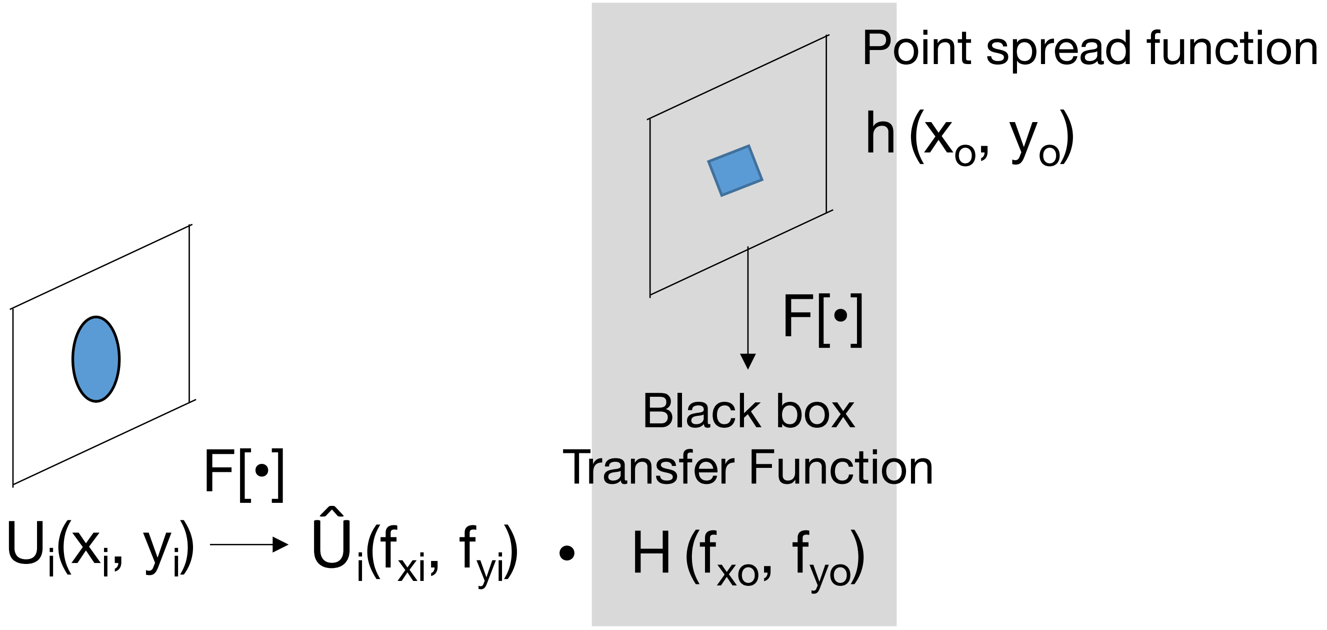

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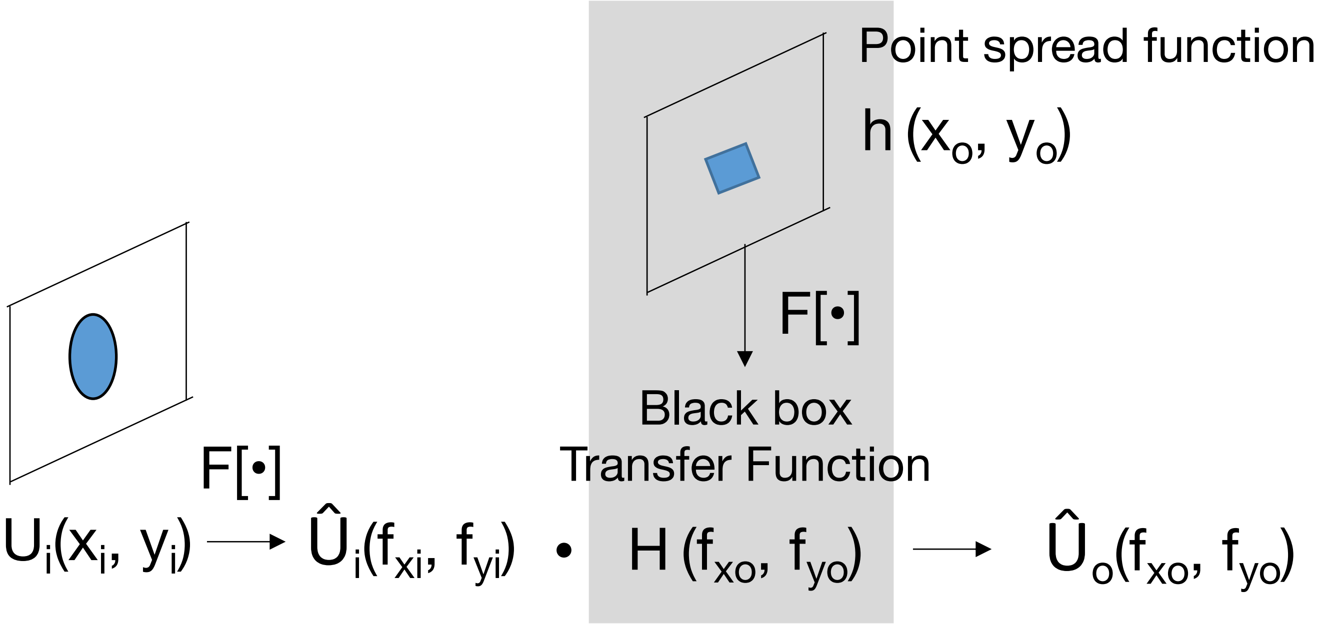

$$\hat{U}_o(f_x, f_y) = \hat{U}_i(f_x, f_y)H(f_x, f_y)$$

Can also multiply Fourier transform of input with transfer function H to obtain Fourier transform of output

Review: black box transforms as a convolution

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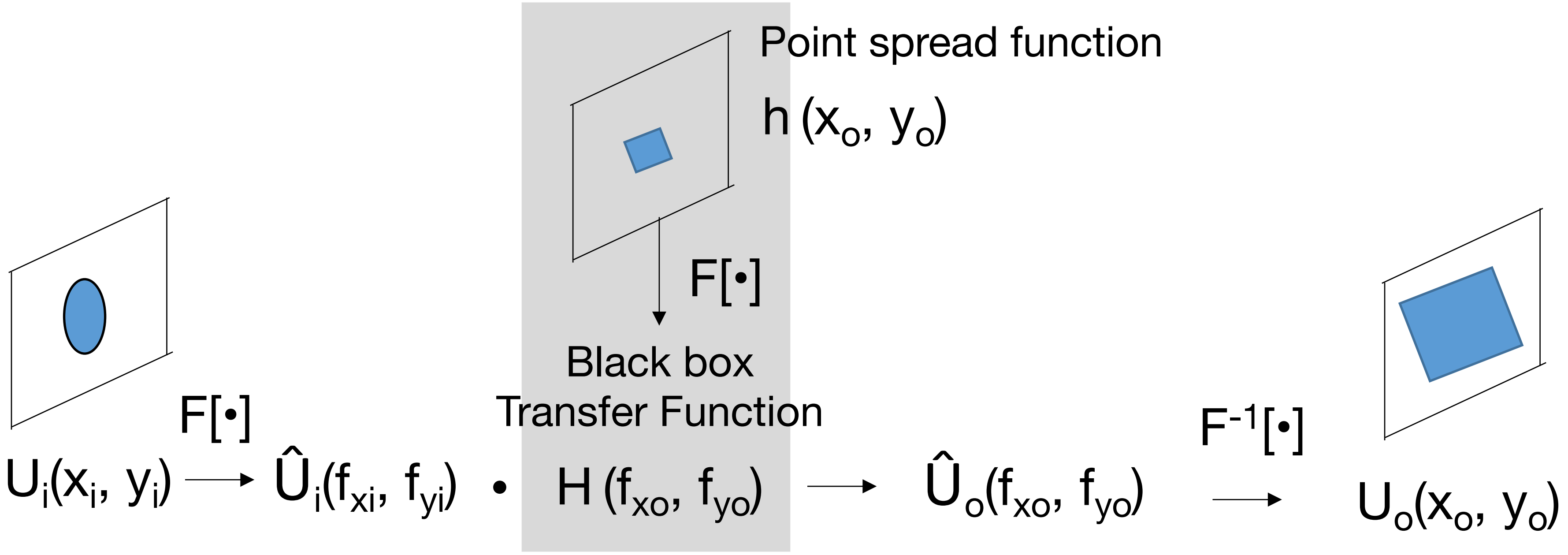

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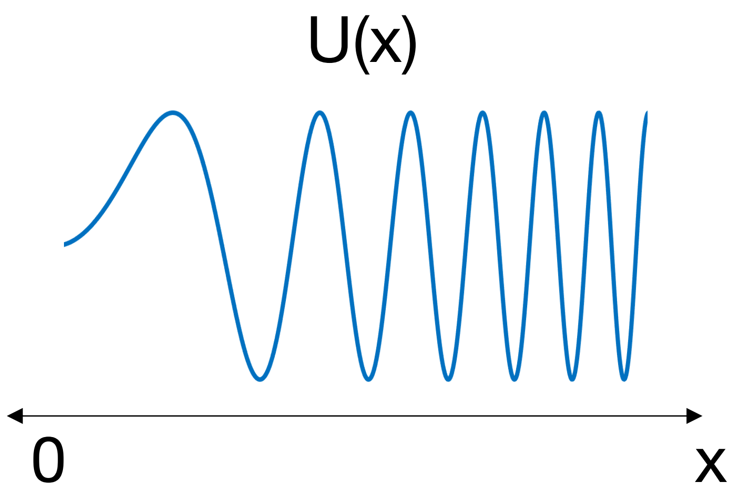
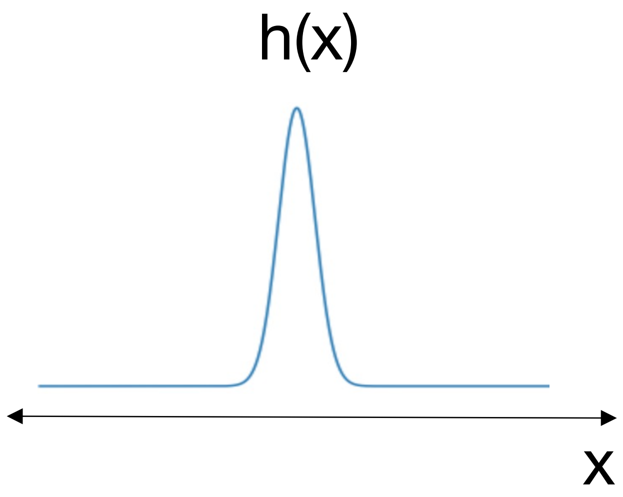
Conceptual questions :

1. Draw what you think the convolution of these two functions looks like:





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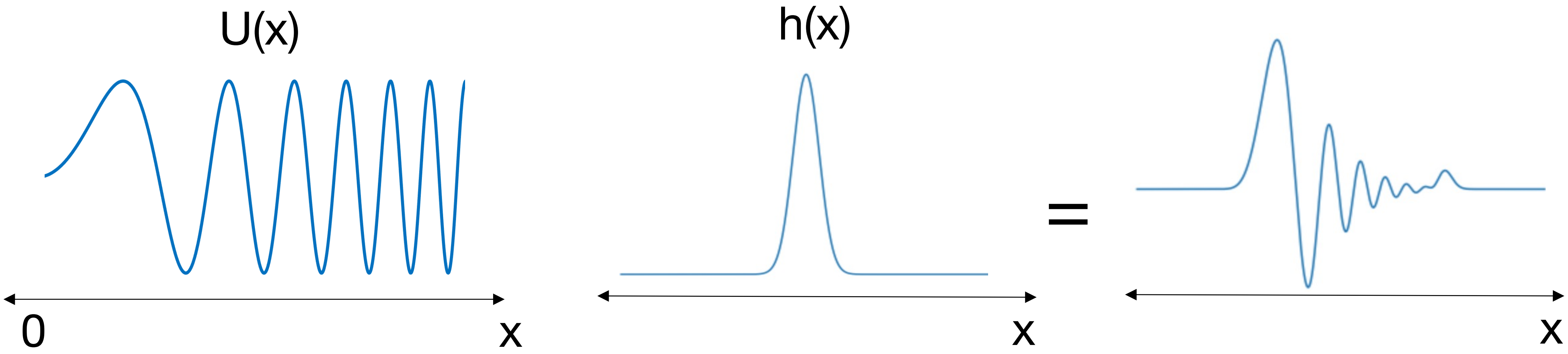

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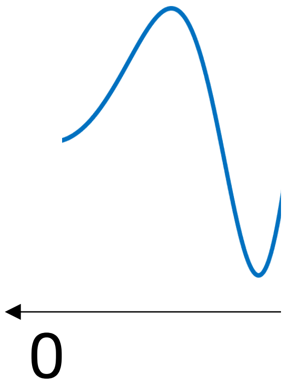
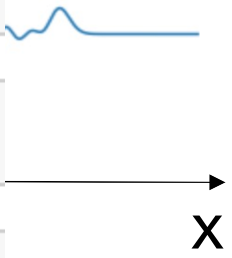
1. Dr




```
In [1]:  import numpy as np

In [50]: x = np.linspace(0,1,1000)

In [72]: U=np.sin(40*np.square(x))
         V=np.exp(-300*np.square(x-0.5))

In [73]: W=np.convolve(U,V)

In [74]: import matplotlib.pylab as plt

In [75]: plt.plot(np.linspace(0,1,1999),W)
Out[75]: [<matplotlib.lines.Line2D at 0x108f42828>]

In [76]: plt.show()
```



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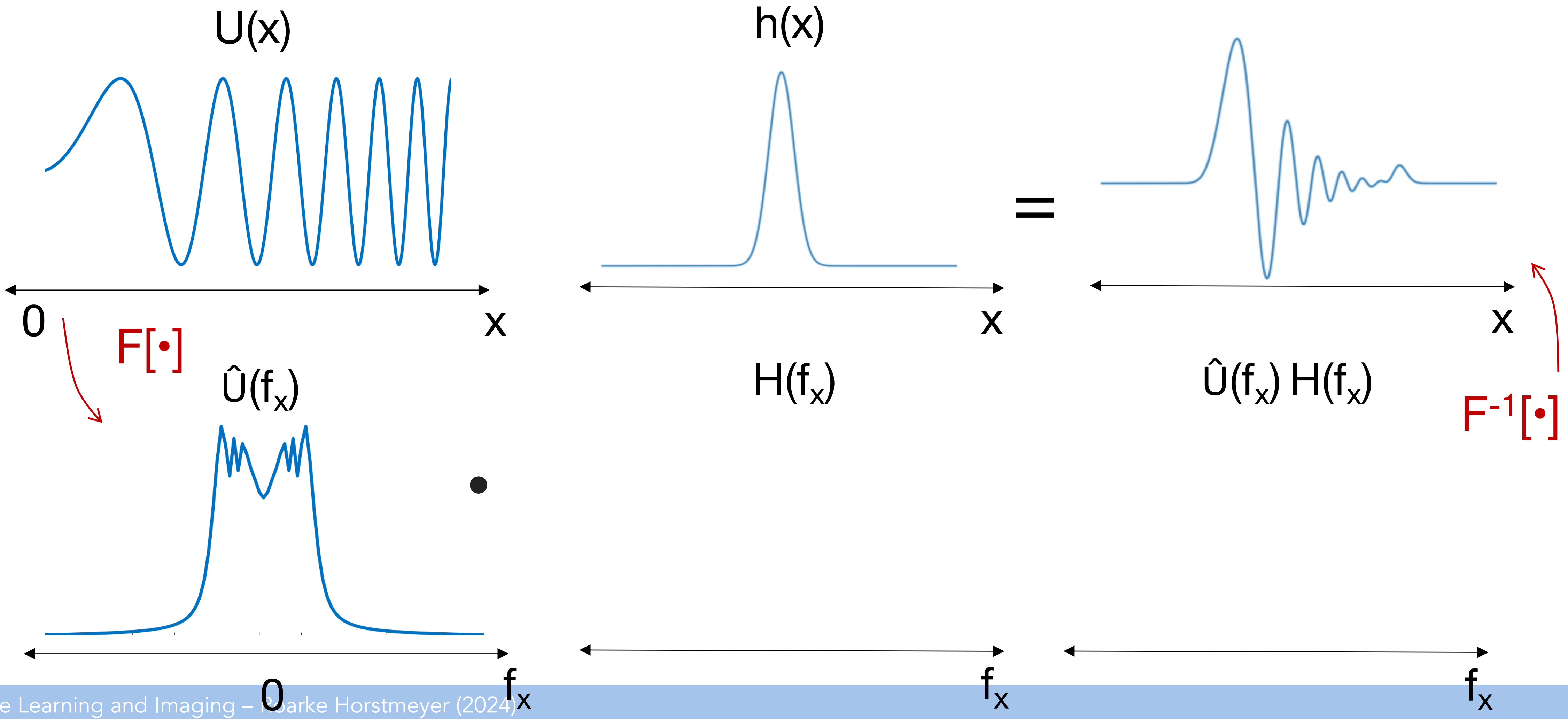

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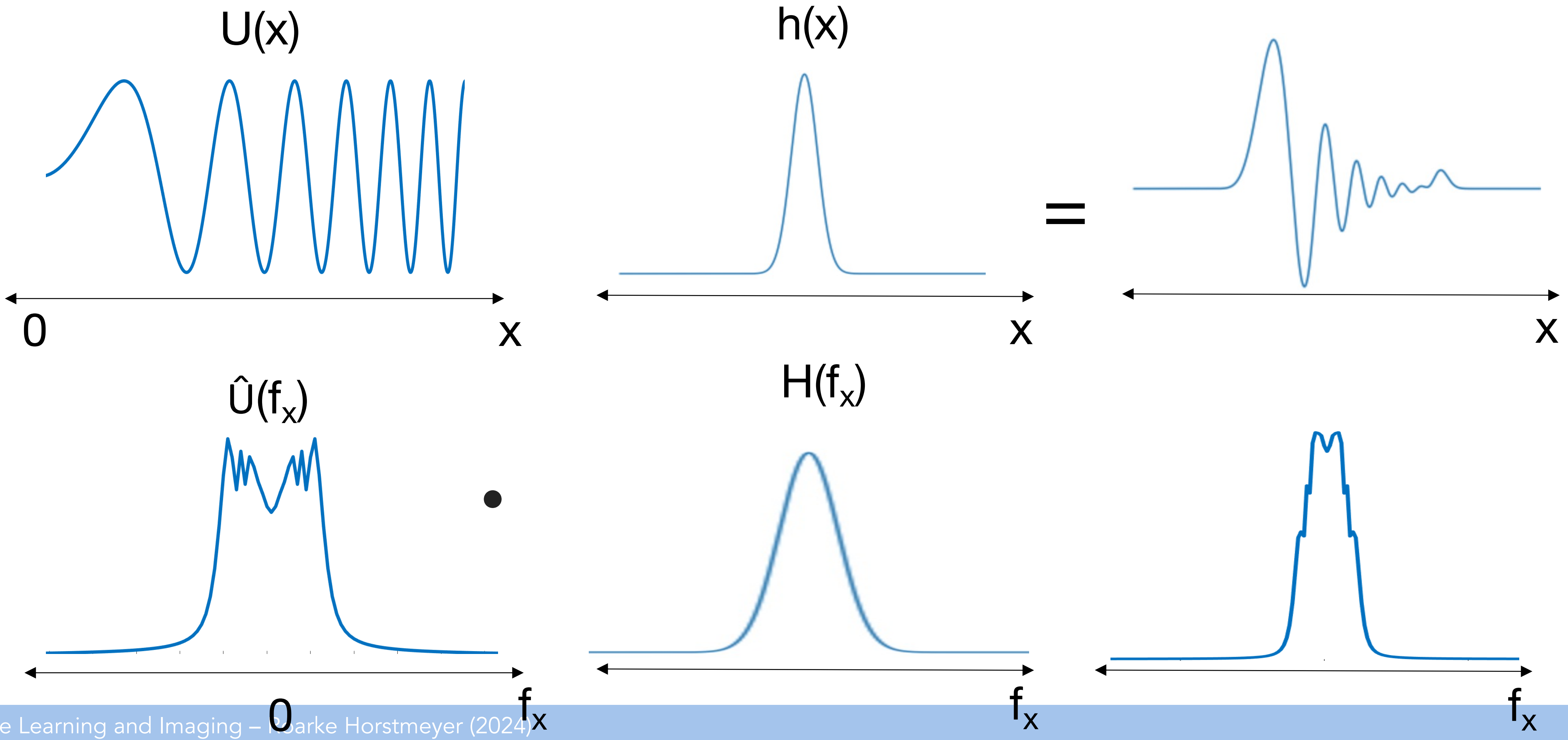
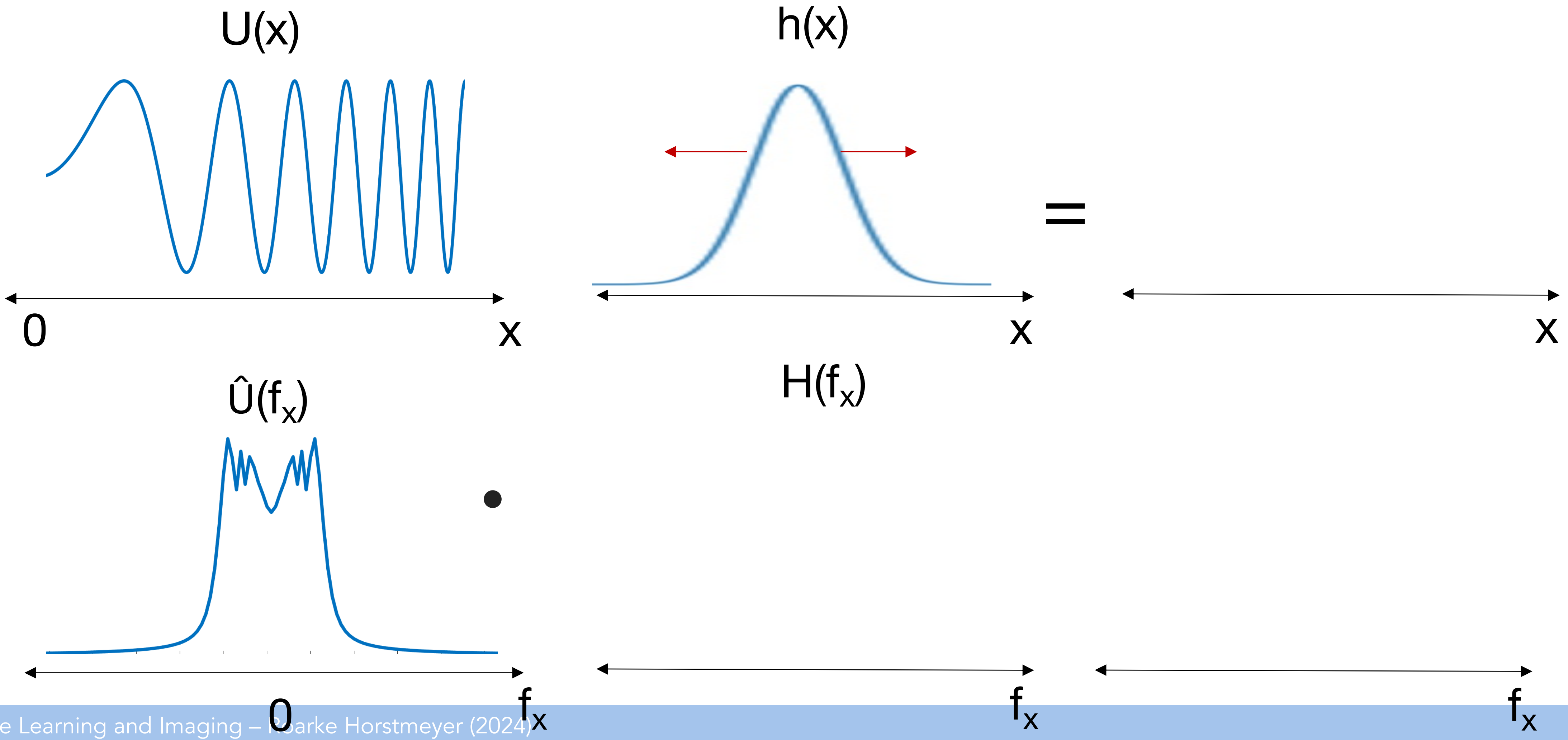


Conceptual questions:

2. Repeat with wider convolution filter:



Conceptual questions:

deep imaging

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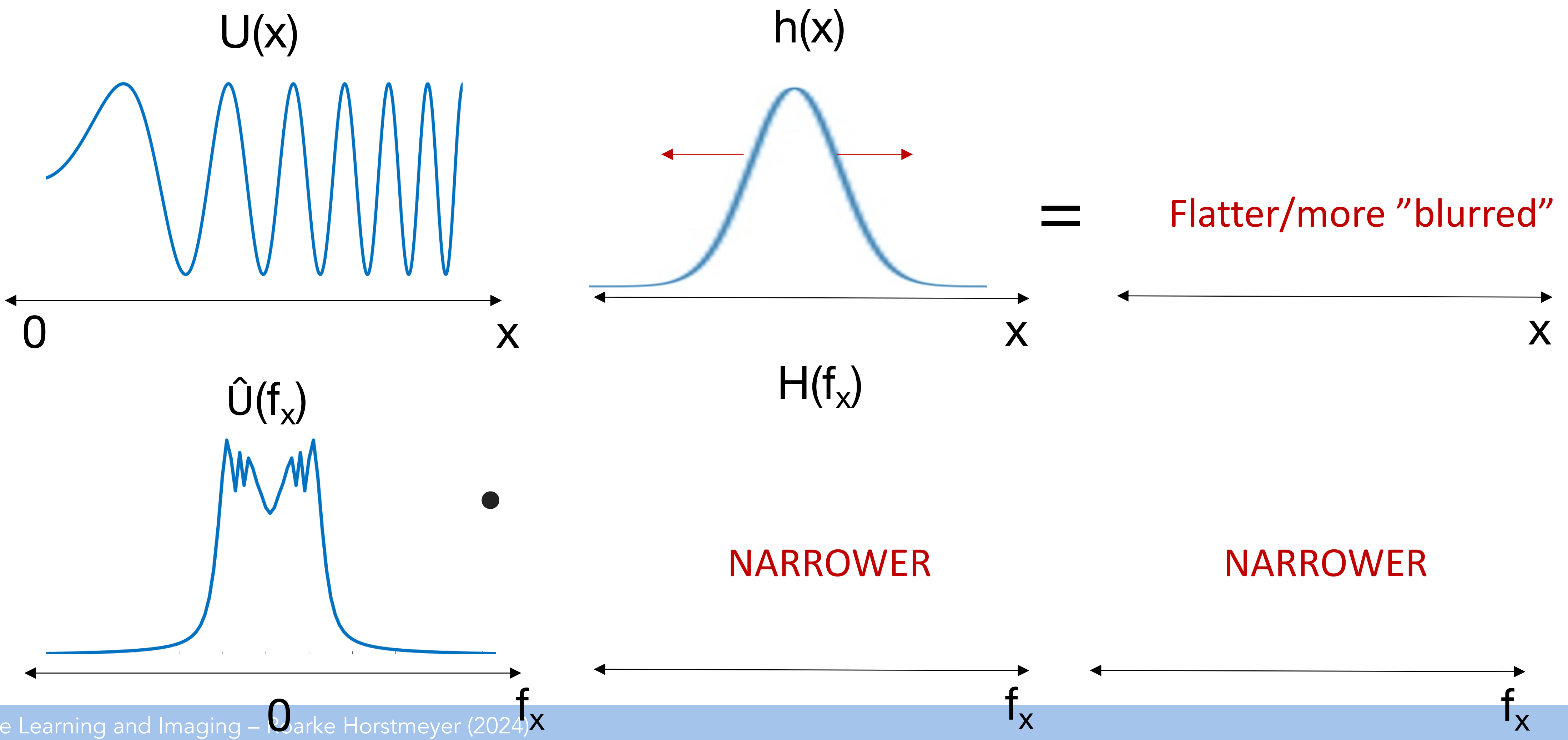
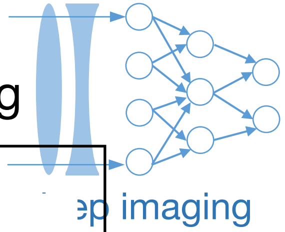
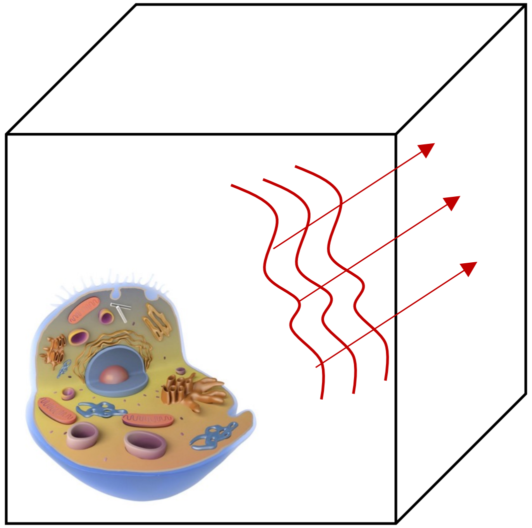


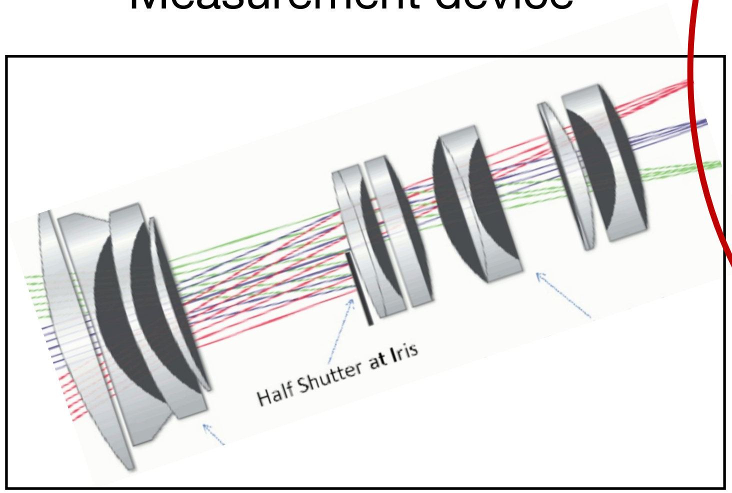
ML+Imaging pipeline introduction



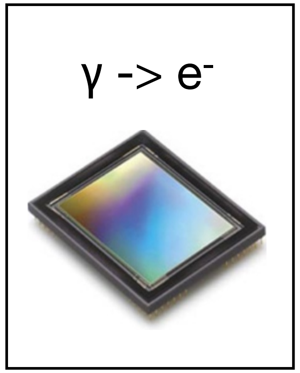
Real World



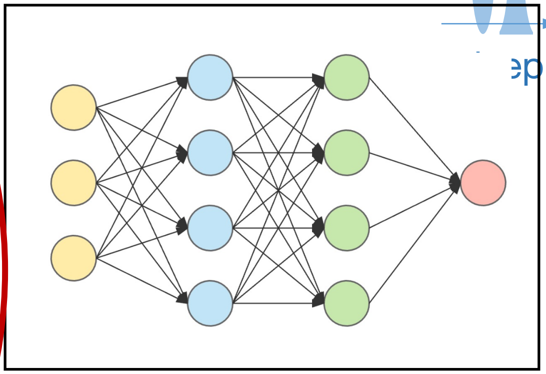
Measurement device



Digitization

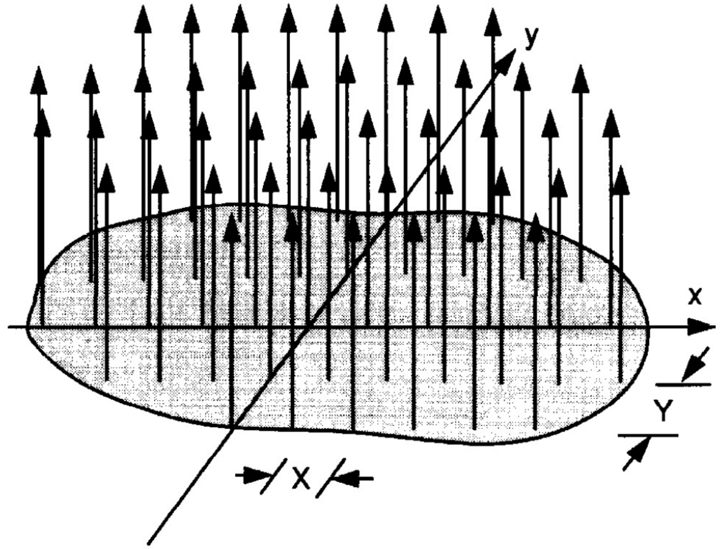


Machine Learning



The Sampling Theorem – from Goodman Section 2.4.1

$$U_s(x, y) = \text{comb}(x/X)\text{comb}(y/Y)U(x, y)$$



Signal sampling occurs with:

- CMOS (pixel) sensors, PMTs, SPADs
- A-to-D after antennas
- A-to-D after acoustic transducers

Sampling interval width X and Y

The Sampling Theorem – from Goodman Section 2.4.1

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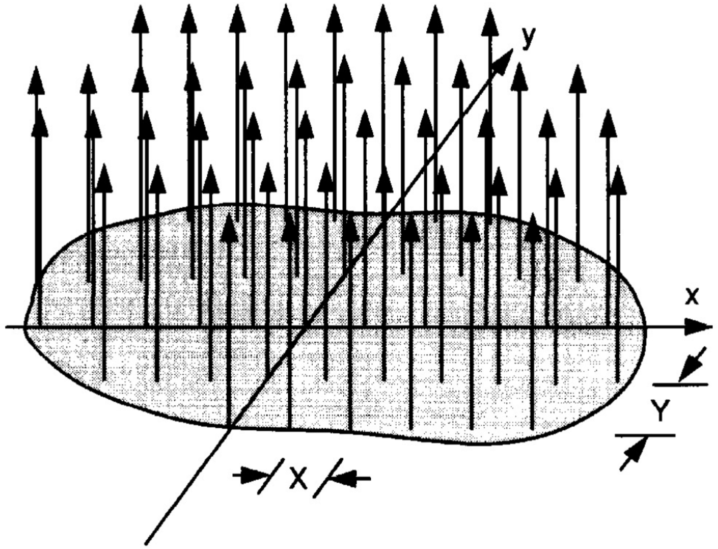

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- CMOS (pixel) sensors, PMTs, SPADs
- A-to-D after antennas
- A-to-D after acoustic transducers

Sampling interval width X and Y

$$\hat{U}_s(f_x, f_y) = \mathcal{F}\left[\mathrm{comb}(x/X)\mathrm{comb}(y/Y)\right] * \hat{U}(f_x, f_y)$$

The Sampling Theorem – from Goodman Section 2.4.1

$$\hat{U}_s(f_x, f_y) = \mathcal{F}\left[\text{comb}(x/X)\text{comb}(y/Y)\right] * \hat{U}(f_x, f_y)$$

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$$\hat{U}_s(f_x, f_y) = \mathcal{F}\left[\text{comb}(x/X)\text{comb}(y/Y)\right] * \hat{U}(f_x, f_y)$$

$$\mathcal{F}\left[\text{comb}(x/X)\text{comb}(y/Y)\right] = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$

The Sampling Theorem – from Goodman Section 2.4.1

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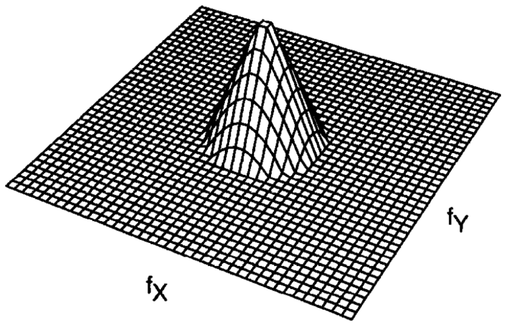
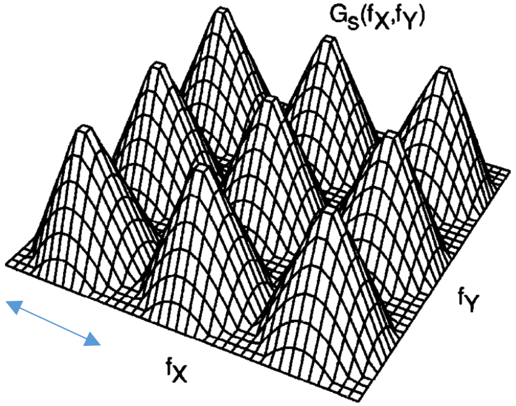


(a)



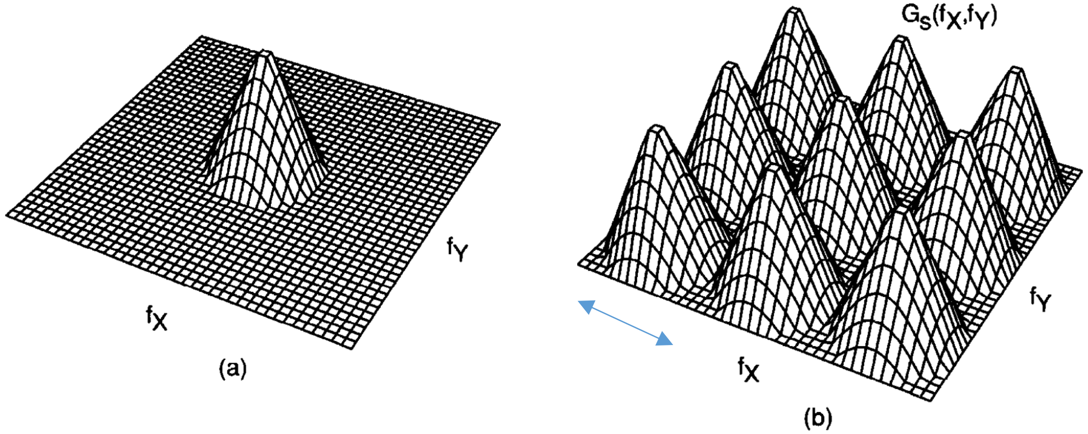
(b)

Signal extends from $(-B_x, -B_y)$ to (B_x, B_y) in Fourier domain

The Sampling Theorem – from Goodman Section 2.4.1

$$\mathcal{F}\left[\text{comb}(x/X)\text{comb}(y/Y)\right] = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$

$$\hat{U}_s(f_x, f_y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\hat{U}\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$



Signal extends from ($-B_x$, $-B_y$) to (B_x, B_y) in Fourier domain

Mask out copies with a rect function:

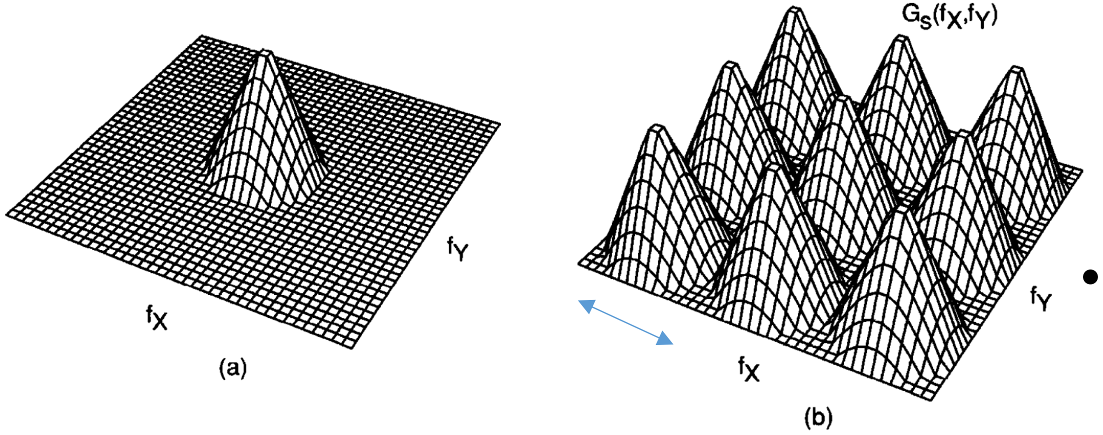
$$\text{rect}\left(\frac{f_x}{2B_x}\right)\text{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

Bandwidth (B_x, B_y) of signal

The Sampling Theorem – from Goodman Section 2.4.1

$$\mathcal{F}\left[\mathrm{comb}(x/X)\mathrm{comb}(y/Y)\right] = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\delta\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$

$$\hat{U}_s(f_x, f_y) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\hat{U}\left(f_x - \frac{n}{X}, f_y - \frac{m}{Y}\right)$$



Signal extends from ($-B_x$, $-B_y$) to (B_x, B_y) in Fourier domain

•

Mask out copies with a rect function:

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F\,[\bullet]$ $F\,[\bullet]$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F[\bullet]$ ↓

$$U(x, y)\mathrm{comb}(x/X)\mathrm{comb}(y/Y)$$

$$\mathrm{rect}\left(\frac{f_x}{2B_x}\right)\mathrm{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F[\bullet]$

$$h(x,y) = 4B_xB_y\mathrm{sinc}(2B_xx)\mathrm{sinc}(2B_yy)$$

deep imaging

$$\text{rect}\left(\frac{f_x}{2B_x}\right)\text{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

$F[\bullet]$

$$h(x,y) = 4B_xB_y\text{sinc}(2B_xx)\text{sinc}(2B_yy)$$

$$h(x,y) * (U(x,y)\text{comb}(x/X)\text{comb}(y/Y)) = U(x,y)$$

$$\text{rect}\left(\frac{f_x}{2B_x}\right)\text{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

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$$h(x,y) = 4B_xB_y\text{sinc}(2B_xx)\text{sinc}(2B_yy)$$

$$h(x,y) * (U(x,y)\text{comb}(x/X)\text{comb}(y/Y)) = U(x,y)$$

$$U(x,y)\text{comb}(x/X)\text{comb}(y/Y) = XY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX,mY)\delta(x-nX,y-mY)$$

$$\text{rect}\left(\frac{f_x}{2B_x}\right)\text{rect}\left(\frac{f_y}{2B_y}\right)\hat{U}_s(f_x, f_y) = \hat{U}(f_x, f_y)$$

F [•] →

$$h(x,y) = 4B_xB_y\text{sinc}(2B_xx)\text{sinc}(2B_yy)$$

$$h(x,y) * (U(x,y)\text{comb}(x/X)\text{comb}(y/Y)) = U(x,y)$$

$U_s(x,y)$ (from beginning) =

$$U(x,y)\text{comb}(x/X)\text{comb}(y/Y) = XY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX,mY)\delta(x-nX,y-mY)$$

$$U(x,y) = 4B_xB_yXY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX,mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

The Sampling Theorem

When sampled appropriately, a discrete signal can *exactly* reproduce a continuous signal:

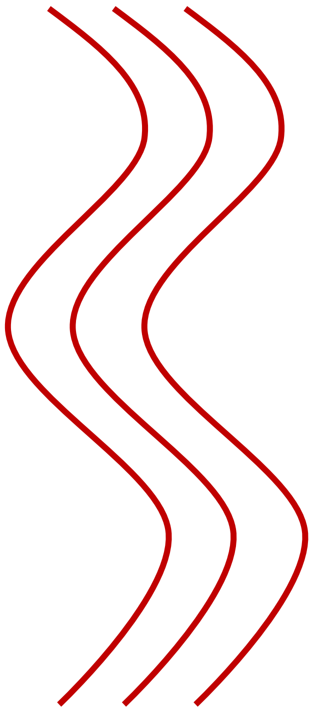
$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX,mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

Continuous signal:
- EM field
- Sound wave
- MR signal

Discretized signal:
- Detected EM field
- Sampled sound wave
- Sampled MR signal

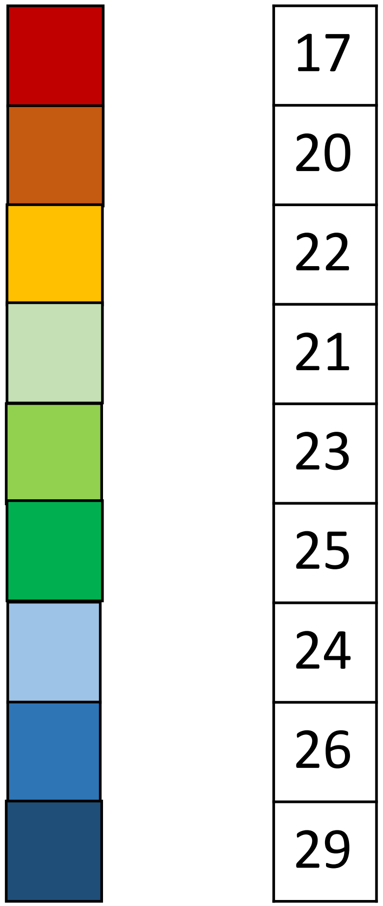
What does the Sampling Theorem mean for us?

Continuous fields



(*) Under certain conditions

Discretize vectors (and matrices)



Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

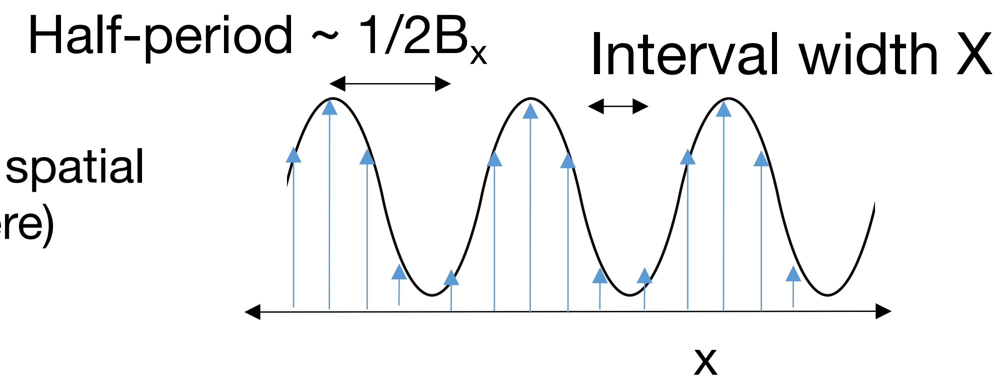
- Sampling must be proportional to bandwidth ($2B_x$ and $2B_y$)
  - “Nyquist” sampling: $X = 1/2B_x$, $Y = 1/2B_y$

Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_x B_y XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY)\text{sinc}\left[2B_x(x - nX)\right] \text{sinc}\left[2B_y(y - mY)\right]$$

- Sampling must be proportional to bandwidth ($2B_x$ and $2B_y$)
  - "Nyquist" sampling: $X = 1/2B_x$, $Y = 1/2B_y$

(Assume largest spatial freq. in signal here)



U(x)comb(x/X)

$X < 1/2B_x$ ✔

Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX,mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

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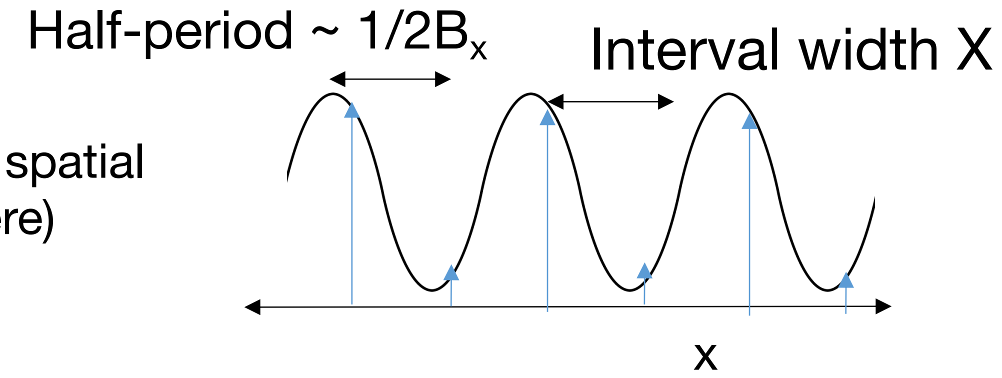


U(x)comb(x/X)

$X = 1/2B_x$ ✔

Nyquist sampling – still sampling peak and trough

Conditions to safely apply the sampling theorem

deep imaging

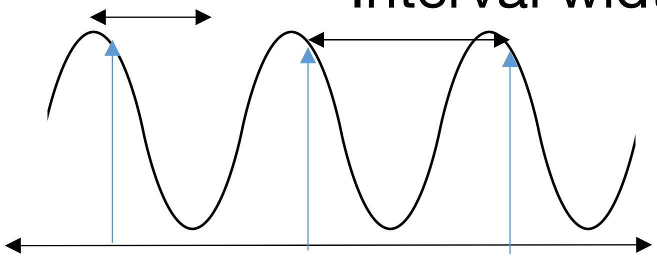
$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY)\text{sinc}\left[2B_x(x-nX)\right] \text{sinc}\left[2B_y(y-mY)\right]$$

- Sampling must be proportional to bandwidth ($2B_x$ and $2B_y$)
  - “Nyquist” sampling: $X = 1/2B_x$, $Y = 1/2B_y$

(Assume largest spatial freq. in signal here)

Half-period ~ $1/2B_x$

Interval width X



x

U(x)comb(x/X)

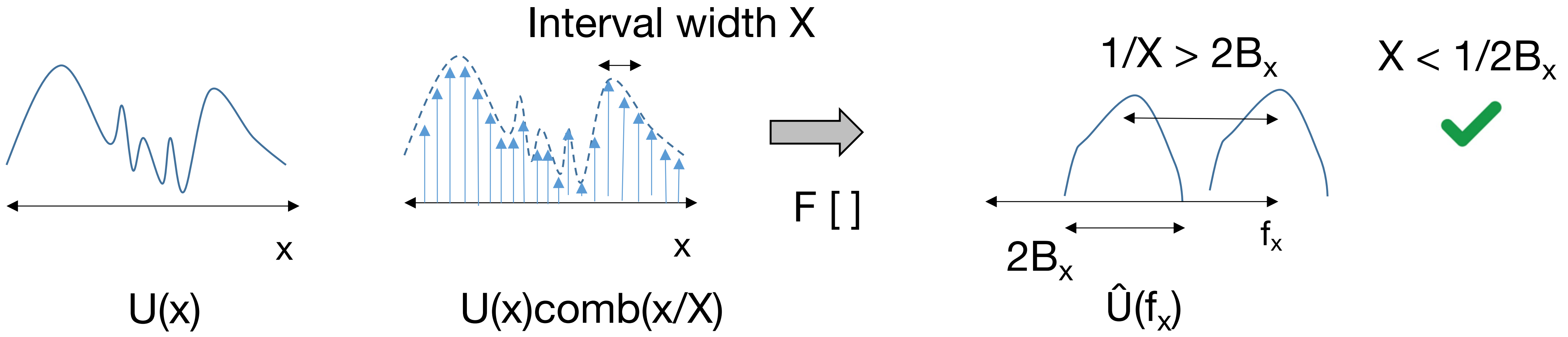
$X > 1/2B_x$

Can’t detect the frequency anymore!

Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_xB_yXY\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}U(nX,mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

- Sampling must be proportional to bandwidth ($2B_x$ and $2B_y$)
  - "Nyquist" sampling: $X = 1/2B_x$, $Y = 1/2B_y$
  - Needed to avoid aliasing



Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX,mY)\text{sinc}\left[2B_x(x-nX)\right]\text{sinc}\left[2B_y(y-mY)\right]$$

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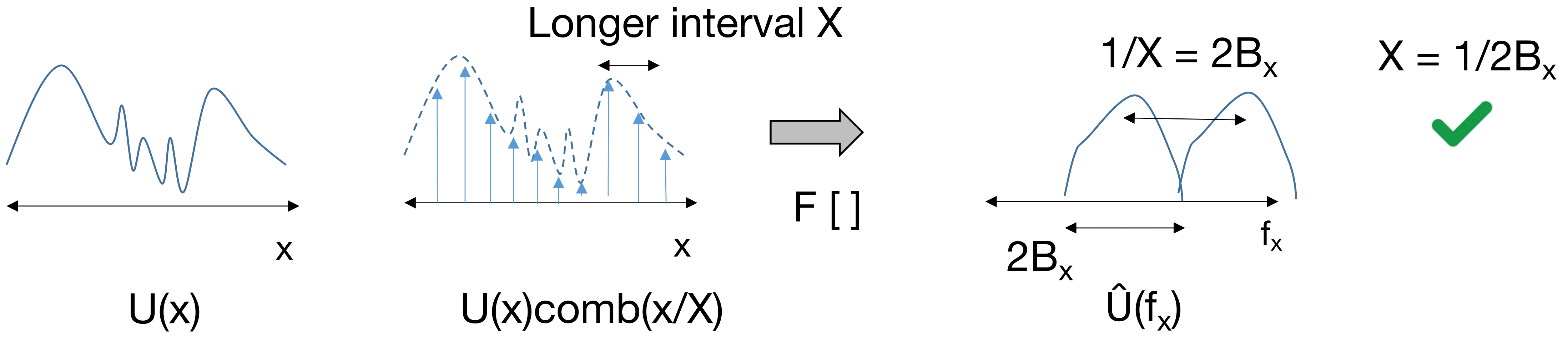

Conditions to safely apply the sampling theorem

$$U(x,y) = 4B_xB_yXY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} U(nX, mY)\mathrm{sinc}\left[2B_x(x-nX)\right]\mathrm{sinc}\left[2B_y(y-mY)\right]$$

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