Lecture 3: From continuous to discrete functions

Machine Learning and Imaging
BME 590L
Roarke Horstmeyer

• Linear black-box systems
• Convolutions in 1D and 2D
• Fourier transforms
• Convolution theorem
• Sampling theorem
ML+Imaging pipeline introduction

Real World

Measurement device

Digitization

$\gamma \rightarrow e^-$

Machine Learning
ML+Imaging pipeline introduction

Real World

Continuous complex fields

(last class)

Black box transformations
- Convolution
- Fourier Transform

(last class, this class)

Digitization

\[ \gamma \rightarrow e^- \]

Machine Learning

Measurement device

Real World

Real World

Real World
ML+Imaging pipeline introduction

Real World

Measurement device

Digitization

Machine Learning

γ -> e⁻

(last class) → (last class, this class) → (this class, next class)
Linear systems and the black box

The “optical” black box system:

An optical black box system maps an input function $U_i(x_i, y_i)$ to an output function $U_o(x_o, y_o)$ via a transform $T$:

$$U_o(x_o, y_o) = T[U_i(x_i, y_i)]$$

Where $T[ ]$ denotes the optical black box transformation
Linear systems and the black box

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Important properties of linear systems:

1. Homogeneity and additivity (superposition):

$$T [aU_1(x,y) + bU_2(x,y)] = aT [U_1(x,y)] + bT [U_2(x,y)]$$
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$$T [aU_1(x,y) + bU_2(x,y)] = aT [U_1(x,y)] + bT [U_2(x,y)]$$

2. Shift invariance: for shift distances $d_x$ and $d_y$, we assume that,

$$U_o(x_o - d_x, y_o - d_y) = T [U_i(x_i-d_x, y_i-d_y)]$$
Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

\[ \delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \]

Input Dirac delta function into the black box:
Black box transforms as a convolution

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Input Dirac delta function into the black box:

$$\mathcal{h}(x_0, y_0) = T [ \partial(x_i, y_i) ]$$
Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

$$\partial(x_i-d_x,y_i-d_y)$$

We know the system is shift invariant:

$$h(x_o-d_y,y_o-d_y) = T [ \partial(x_i-d_x,y_i-d_y) ]$$
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A “perfect” point source

\[ \partial(x_i, y_i) \rightarrow \text{LSI system} \rightarrow h(x_o, y_o) \]

Point-spread function

\[ h(x_o, y_o) = T [ \partial(x_i, y_i) ] \]

\( h(x_o, y_o) \) is the system’s point-spread function.
Knowing the point-spread function, it is direct to model any output of the black box, given an input:

\[
U_o(x_o, y_o) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_i(x_i, y_i) h(x_o - x_i, y_o - y_i) \, dx_i \, dy_i
\]
1D convolution example

Steps to perform a convolution:

1. Flip one signal (the second one = the PSF)
2. Position PSF right before overlap

With incremental steps:

3. Step PSF over to position $x_o$
4. Compute *area* of overlap of two functions
5. Convolution value at $x_o$ = area of overlap
6. Repeat 3-5 until signals do not overlap

https://en.wikipedia.org/wiki/Convolution
2D convolution example

- Direct extension of 1D concept to 2D functions
- Note – it is effectively the same with discrete functions = matrices

\[ U_1(x, y) \times U_0(x, y) = U(x, y) \]
2D convolution example

High-res. real-world object

$U_1(x,y)$

Blur caused by camera lens

$x \ast y \ast y^2 = x^2$

Image at camera sensor plane

$U_0(x,y)$
Optical modification Ex. #1: The cubic phase mask

- Standard camera
- Point-spread function

in focus
Optical modification Ex. #1: The cubic phase mask

Standard camera:
Limited depth-of-field

in focus
defocused
Optical modification Ex. #1: The cubic phase mask

Standard camera:
Limited depth-of-field

in focus  defocused  defocused
Optical modification Ex. #1: The cubic phase mask

Fourier plane

Blur proportional to Fourier transform of shape of aperture
Optical modification Ex. #1: The cubic phase mask

Standard camera:
Limited depth-of-field

in focus

defocused
defocused

Cubic phase mask:
extended depth-of-field

CPM Phase profile

Optical modification Ex. #1: The cubic phase mask

Standard camera: Limited depth-of-field

Cubic phase mask: extended depth-of-field

in focus
defocused
defocused

Same blur!

CPM Phase profile

Optical modification Ex. #1: The cubic phase mask

Standard camera: Limited depth-of-field

Cubic phase mask: extended depth-of-field

Optical modification Ex. #1: The cubic phase mask

Insert CPM

Uniformly Blurry image

Simple image deblurring (deconv.)

All in-focus image

Optical modification Ex. #1: The cubic phase mask

- Insert CPM

- Uniformly Blurry image

- Simple image deblurring (deconv.)

- All in-focus image

Standard camera image  
CPM image, raw  
CPM image, deconvolved
Optical modification Ex. #1b: Double helix mask

Insert DHM

Depth-varying image

in focus  defocused  defocused

Depth detection

Jia et al., Nature Photonics 2014

Moerner Lab Nobel Prize in Chemistry, 2014
Useful properties of the convolution

1. **Commutativity** \( U(x) * h(x) = h(x) * U(x) \)

\[ \Rightarrow \text{You can choose which signal to “flip”} \]
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2. **Associativity** \( U(x) * [V(x) * W(x)] = [U(x) * V(x)] * W(x) \)
   \( \Rightarrow \) Can change order \( \rightarrow \) sometimes one order is easier than another
Useful properties of the convolution

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2. **Associativity** \( U(x) * [V(x) * W(x)] = [U(x) * V(x)] * W(x) \)
   \( \Rightarrow \) Can change order → sometimes one order is easier than another

3. **Distributivity** \( U(x) * [h_1(x) * h_2(x)] = U(x) * h_1(x) + U(x) * h_2(x) \)
Signals in space and spatial frequency

• What we have so far:
  • Continuous & (possibly) complex function for images across space
  • Black-box linear transformation from one domain to the next via convolution
Signals in space and spatial frequency

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• Analogy:
  • Time-varying voltage/current going through a circuit
  • Audio signal passing through a filter

Complex function of time -> frequency
Signals in space and spatial frequency

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Complex function of time -> frequency

Fourier Transforms
Signals in space and spatial frequency

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- Here, we have 2D (complex) function across space \((x,y) \rightarrow spatial\ frequency \((f_x, f_y)\)
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- Here, we have 2D (complex) function across space \((x,y)\) -> spatial frequency \((f_x, f_y)\)

\[
U(x,y) \xrightarrow{\text{Fourier Transform}} \hat{U}(f_x, f_y)
\]
Continuous Fourier transforms – for 2D images

Decomposition of a signal into elementary functions of form, \( \exp (-2\pi i (f_x x + f_y y)) \):

\[
\mathcal{F}\{U(x, y)\} = \hat{U}(f_x, f_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x, y) \exp (-2\pi i (f_x x + f_y y)) \, dx \, dy
\]
Continuous Fourier transforms – for 2D images

Decomposition of a signal into elementary functions of form, \( \exp \left( -2\pi i(f_x x + f_y y) \right) \): 

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\mathcal{F}\{U(x, y)\} = \hat{U}(f_x, f_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U(x, y) \exp \left( -2\pi i(f_x x + f_y y) \right) \, dx \, dy
\]

\( U \) is absolutely integrable & no infinite discontinuities. The inverse Fourier transform is,

\[
\mathcal{F}^{-1}\{\hat{U}(f_x, f_y)\} = U(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{U}(f_x, f_y) \exp \left( 2\pi i(f_x x + f_y y) \right) \, df_x \, df_y
\]

Additional Details:
- Goodman Chapter 2.1
- Mathworld/Wikipedia, Fourier Transform
A few examples of Fourier transform pairs, 1D

\[ U(x) \quad \text{and} \quad \hat{U}(f_x) \]
Examples of Fourier transform pairs, 2D

(a) $\text{circ}(r)$

(b) $J_1(2\pi p)/p$
Deep imaging

\[ U_1(x,y) \quad \text{Cheetah} \]

\[ U_2(x,y) \quad \text{Zebra} \]
\[ \hat{U}_1(f_x, f_y) \]

**magnitude of cheetah**

**phase of cheetah**

\[ \hat{U}_2(f_x, f_y) \]

**magnitude of zebra**

**phase of zebra**
Important properties of the Fourier transform

- Linearity
- Scaling
- Shift
- Parseval’s Theorem (energy conservation)
- Fourier integral theorem

Additional Details:
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Convolution - Fourier Transform relationship: Convolution Theorem

Convolution theorem. If $\mathcal{F}\{g(x, y)\} = G(f_x, f_y)$ and $\mathcal{F}\{h(x, y)\} = H(f_x, f_y)$, then

$$\mathcal{F}\left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi, \eta) h(x - \xi, y - \eta) \, d\xi \, d\eta \right\} = G(f_x, f_y) H(f_x, f_y).$$

“The convolution of two functions in space can be performed by a multiplication in the Fourier domain (spatial frequency domain)”
Example of convolution theorem, 1D

\[ U_1(x) \ast U_2(x) = \text{sinc}(ax) \]
Example of convolution theorem, 1D

\[ U_1(x) \]

\[ U_2(x) = \text{sinc}(ax) \]

\[ \hat{U}_1(f_x) \]

\[ \hat{U}_2(f_x) = \text{rect}(f_x/a) \]

\[ F[U_1] \]

\[ F[U_2] \]
Example of convolution theorem, 1D

\[ U_1(x) \]

\[ U_2(x) = \text{sinc}(ax) \]

\[ F[U_1] \]

\[ F[U_2] \]

\[ \hat{U}_1(f_x) \cdot \hat{U}_2(f_x) = \text{rect}(f_x/a) \]
Example of convolution theorem, 1D

\[ U_1(x) * U_2(x) = F^{-1}[\hat{U}_2 \hat{U}_1] \]

\[ \hat{U}_2(f_x) = \text{rect}(f_x/a) \]

\[ F \{ U_1 \} \]

\[ F \{ U_2 \} \]

\[ F \{ U_1 \} * F \{ U_2 \} = F \{ U_1 U_2 \} \]

\[ \hat{U}_1(f_x) \cdot \hat{U}_2(f_x) = \hat{U}_1(f_x) \hat{U}_2(f_x) \]

\[ \text{rect}(f_x/a) \]
Next Class: The Sampling Theorem – from Goodman Section 2.4.1

\[ U_s(x, y) = \text{comb}(x/X)\text{comb}(y/Y)U(x, y) \]

Signal sampling occurs with:
- CMOS (pixel) sensors, PMTs, SPADs
- A-to-D after antennas
- A-to-D after acoustic transducers

Sampling interval width \( X \) and \( Y \)