deep imaging

# Lecture 2: Mathematical preliminaries for continuous functions

Machine Learning and Imaging

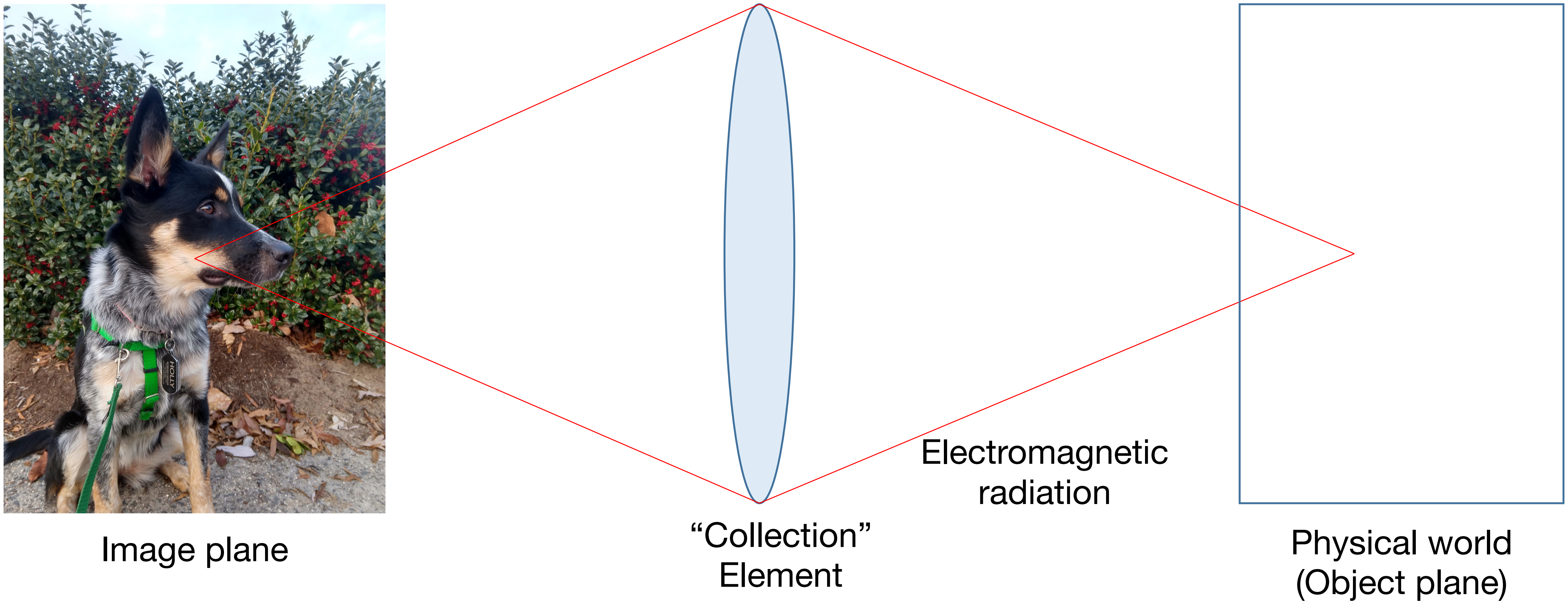
BME 548L

Roarke Horstmeyer

- Light as a continuous wave
- Light transformations as a black box
- Linear black-box systems
- Convolutions in 1D and 2D

## Recall: what is an image?

2. "Physical" Interpretation

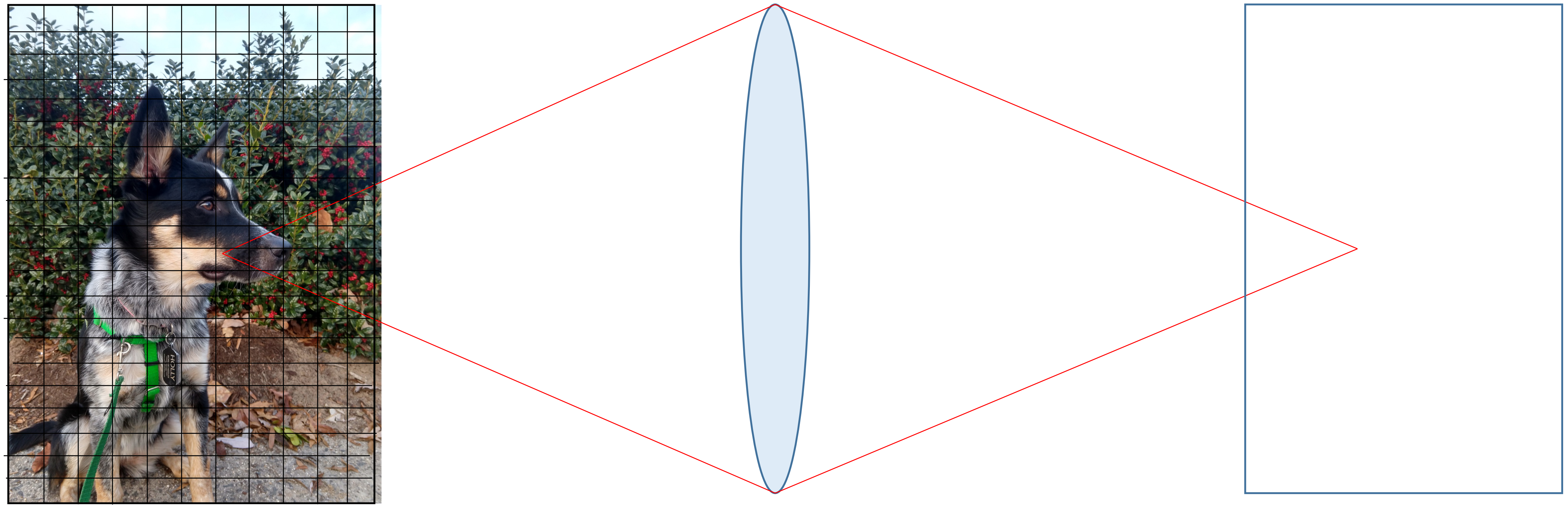



*Continuous signal:* $\boxed{I(x,y),(x,y)\in(-\infty,\infty)}$

# Recall: what is an image?

## 3. "Digital" Interpretation

*n* x *m* array

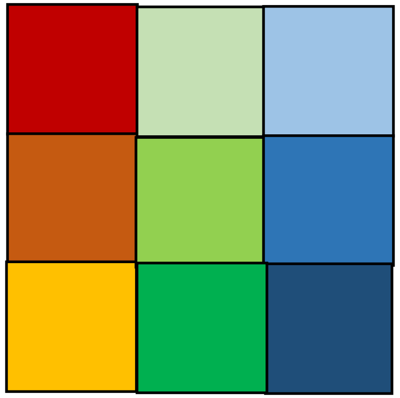


Photons to electrons → Digitazation → *Discrete signal*

$$\boxed{I_s(x, y), (x, y) \in Z^{n \times m}}$$

## Images unrolled into vectors

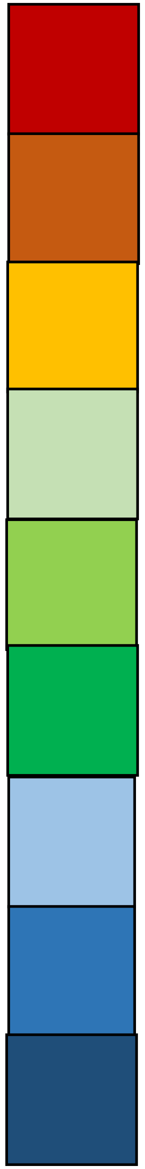
$I_s(x,y)$



| | | |
|---|---|---|
| 17 | 21 | 24 |
| 20 | 23 | 26 |
| 22 | 25 | 29 |

$\mathrm{vec}_c[\bullet]$ $\longrightarrow$

$I_s$



| |
|---|
| 17 |
| 20 |
| 22 |
| 21 |
| 23 |
| 25 |
| 24 |
| 26 |
| 29 |

## Quick break:

A. Please create a 5x5 matrix with integer values that represent a "smiley face"

B. Please transform your matrix from (A) into a column vector

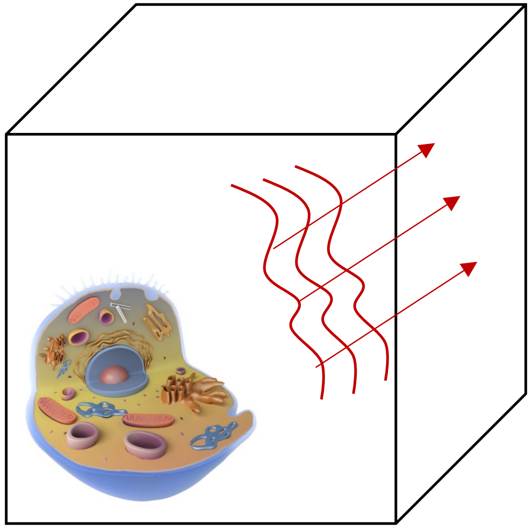
# ML+Imaging pipeline introduction

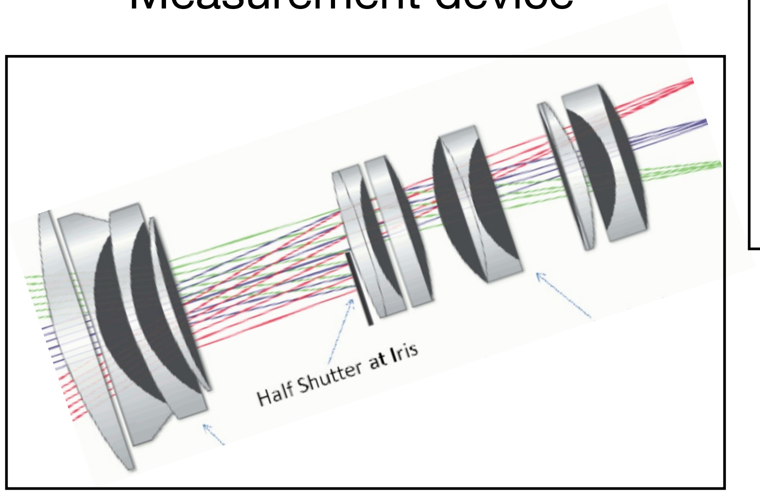
Machine Learning

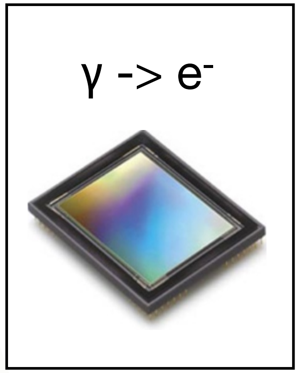
Digitization

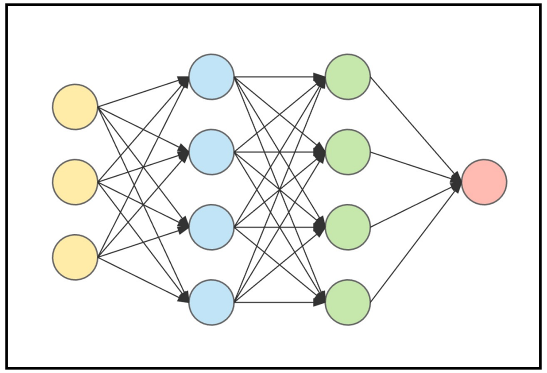
Measurement device

Real World





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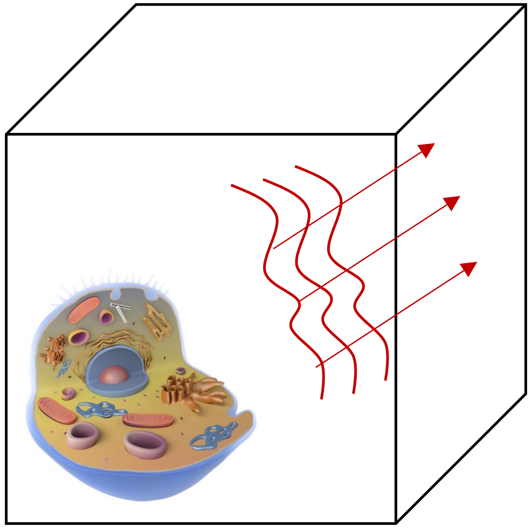
# ML+Imaging pipeline introduction

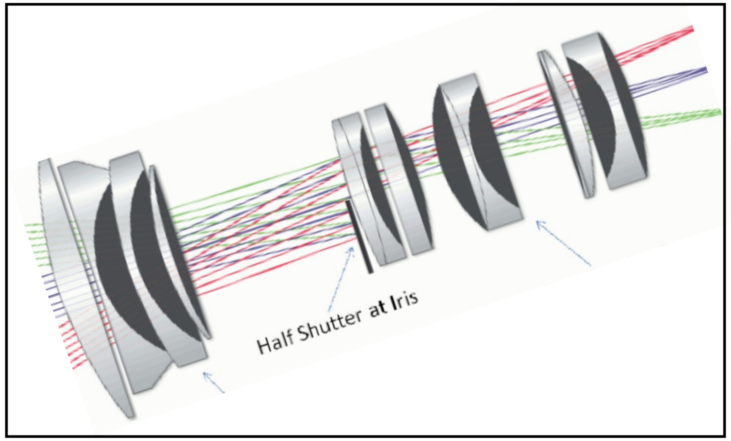
Real World



Continuous
complex fields

(this class)

Measurement device



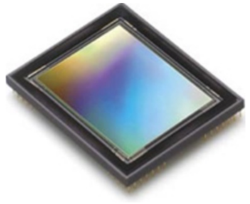

Black box transformations

- Convolution
- Fourier Transform

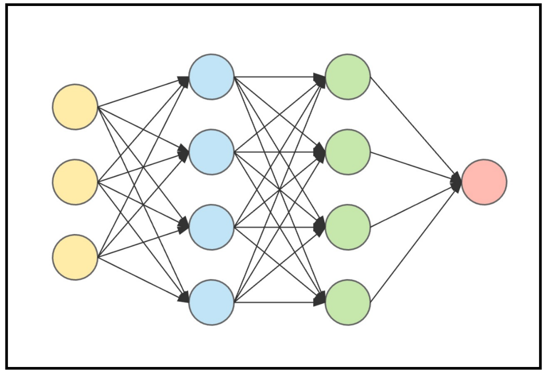
(this class, next class)

Digitization

$\gamma \rightarrow e^-$

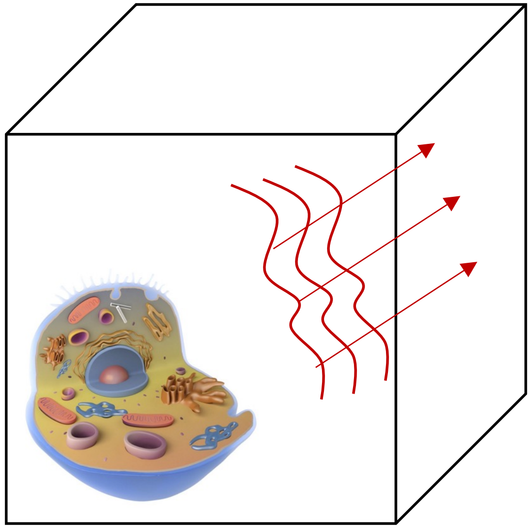


Machine Learning

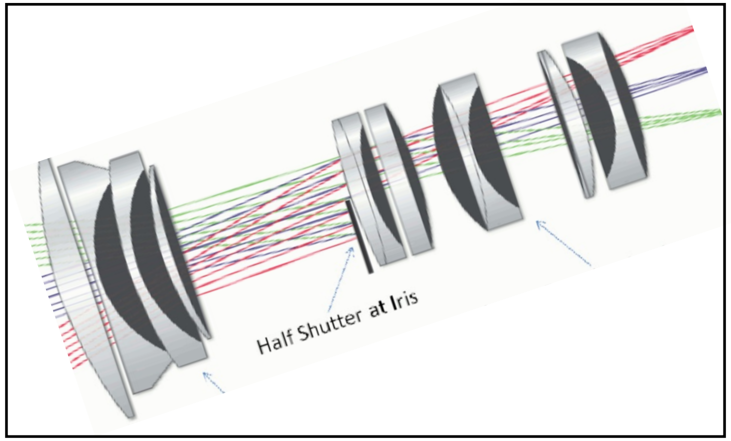


# ML+Imaging pipeline introduction

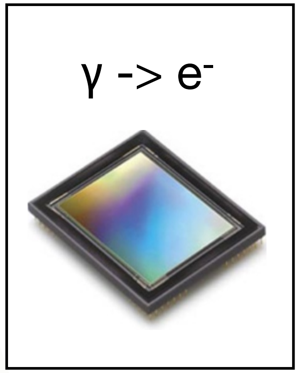
Real World



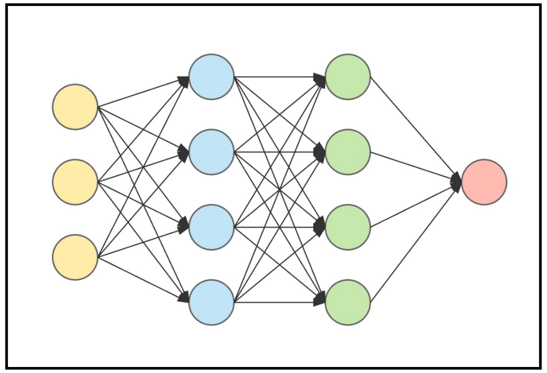
Measurement device




Digitization




Machine Learning



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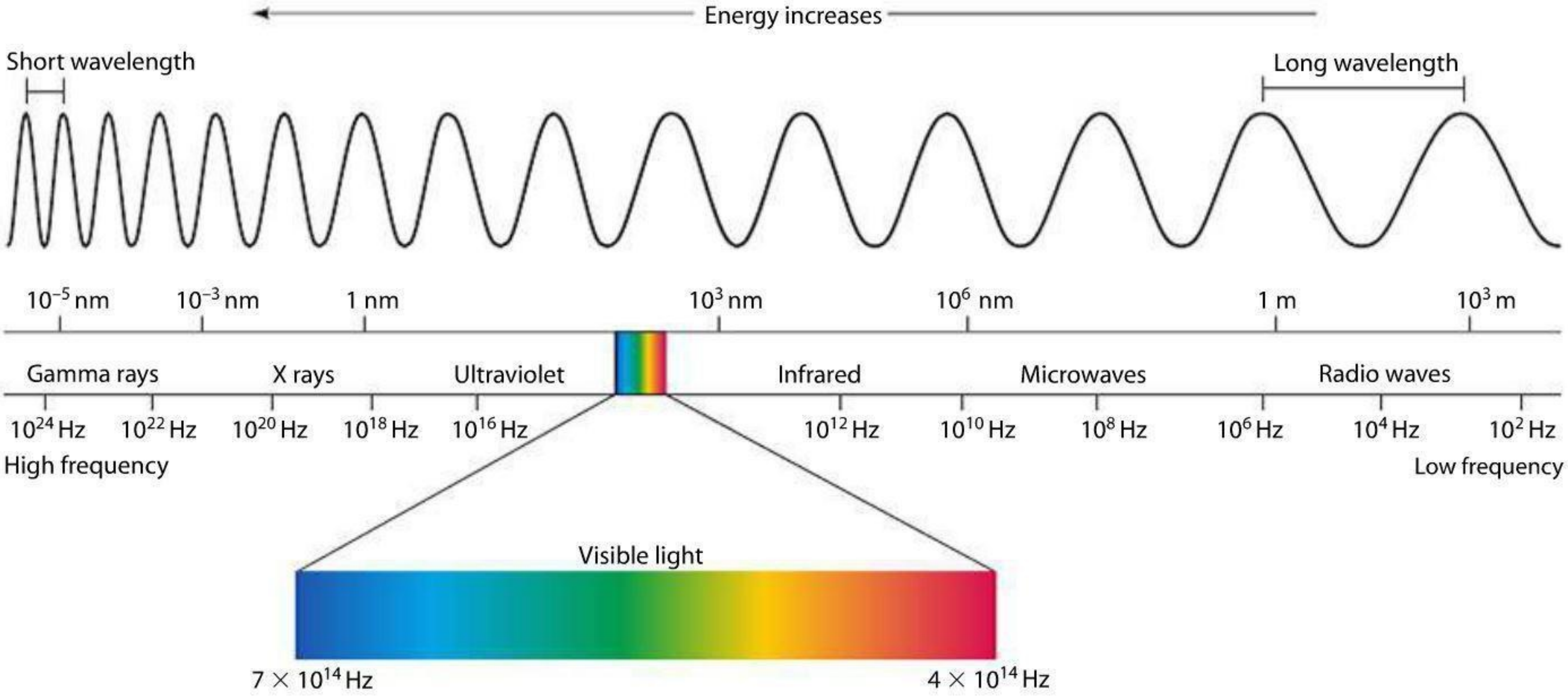
(this class) → (this class, next class) → (next class)

# Start at the beginning: Electromagnetic waves




From: https://www.miniphysics.com/electromagnetic-spectrum_25.html

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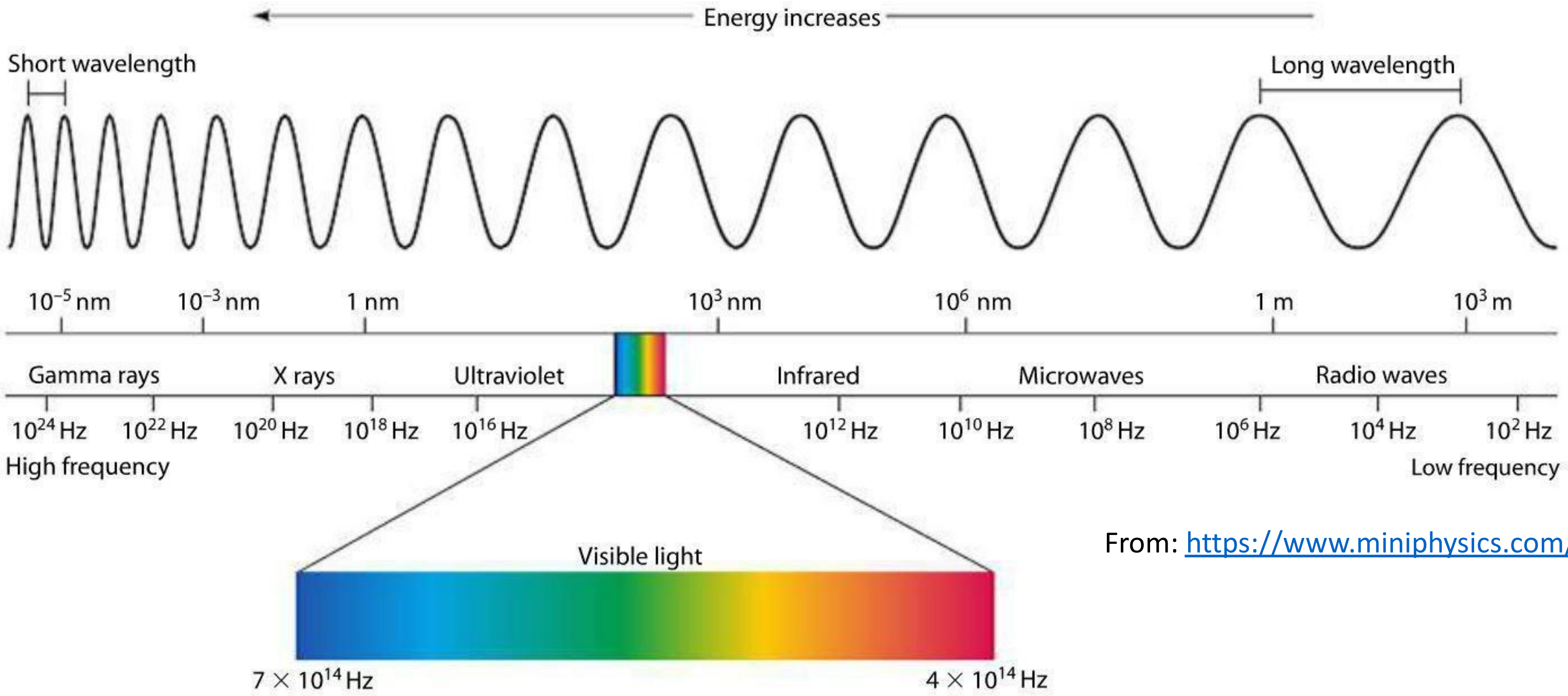


From: https://www.miniphysics.com/electromagnetic-spectrum_25.html

Maxwell's equations

$$
\begin{aligned}
\nabla \times \vec{\mathcal{E}} &= -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t} \\
\nabla \times \vec{\mathcal{H}} &= \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \\
\nabla \cdot \epsilon \vec{\mathcal{E}} &= 0 \\
\nabla \cdot \mu \vec{\mathcal{H}} &= 0.
\end{aligned}
$$

→

Free-space propagation

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0$$

→

Scalar solution, 1 freq.

$$A(\mathbf{r}_1) \cos(\mathbf{k}\mathbf{r}_1 - \omega t)$$

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## Start at the beginning: EM fields and the black box

The general idea:

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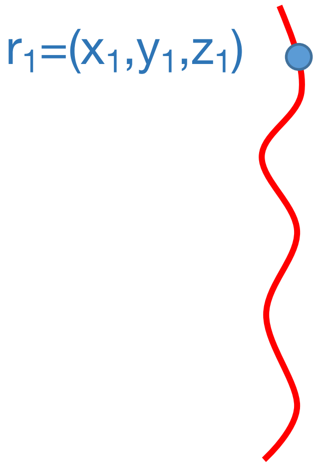
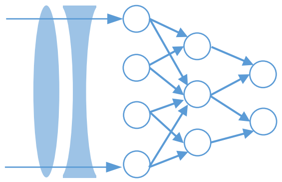
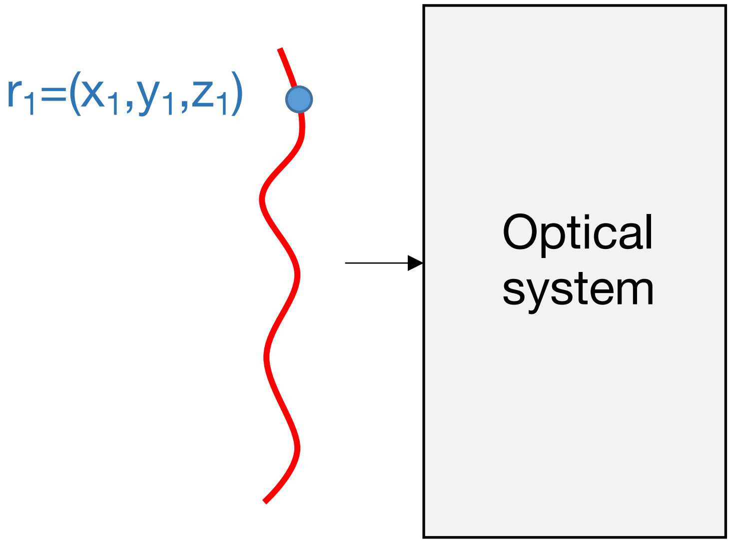



$$U(\mathbf{r_1}) = A(\mathbf{r_1}) \cos(\mathbf{kr_1} - \omega t)$$

(We will get into the details of optical fields in a few weeks)

## Start at the beginning: EM fields and the black box




$$U(\mathbf{r_1}) = A(\mathbf{r_1}) \cos(\mathbf{kr_1} - \omega t)$$

The general idea:

1. We will treat light as a wave (an "optical field")
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties

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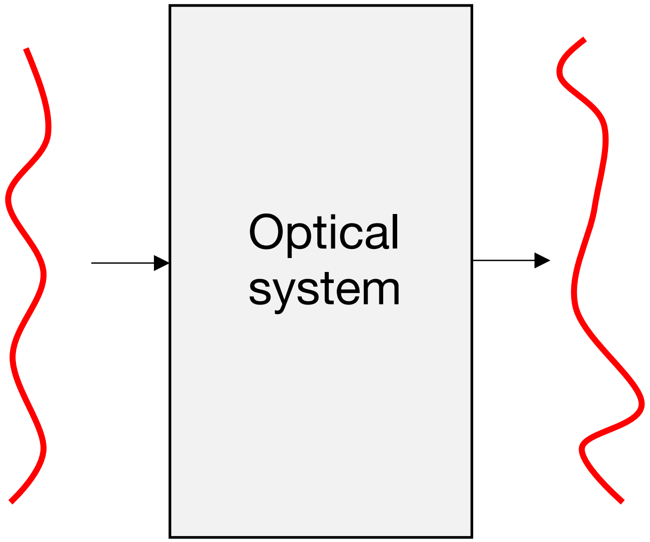


$A(\mathbf{r_1})\cos(\mathbf{kr_1} - \omega t)$ $A(\mathbf{r_2})\cos(\mathbf{kr_2} - \omega t)$

The general idea:

1. We will treat light as a wave (an "optical field")
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties
4. The black box outputs an optical field

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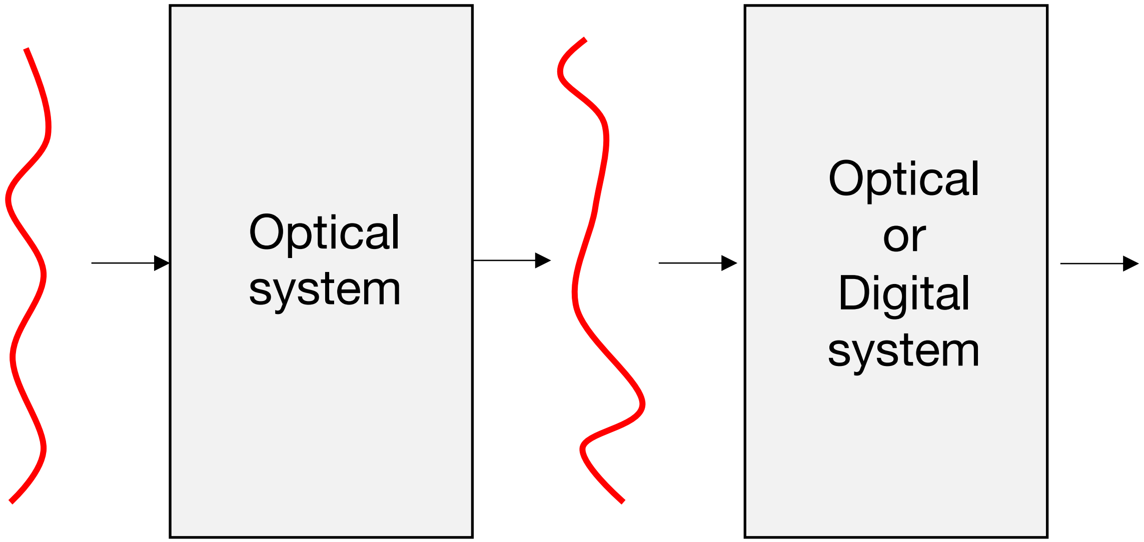



$A(\mathbf{r}_1)\cos(\mathbf{kr}_1 - \omega t)$ $\quad$ $A(\mathbf{r}_2)\cos(\mathbf{kr}_2 - \omega t)$

The general idea:

1. We will treat light as a wave (an "optical field")
2. It enters an optical system, which we treat as a black box
3. This black box has a number of useful properties
4. The black box outputs an optical field, which then enters another optical system or a digital system
5. We can cascade these boxes…

## Linear systems and the black box

**Simplification #1:** Let’s forget about light changing as a function of time. It does so either way too fast, or way too slow:

$$A(\mathbf{r})\cos(\mathbf{kr} - \omega t) \rightarrow A(\mathbf{r})\cos(\mathbf{kr})$$

## Linear systems and the black box

**Simplification #1:** Let's forget about light changing as a function of time. It does so either way too fast, or way too slow:

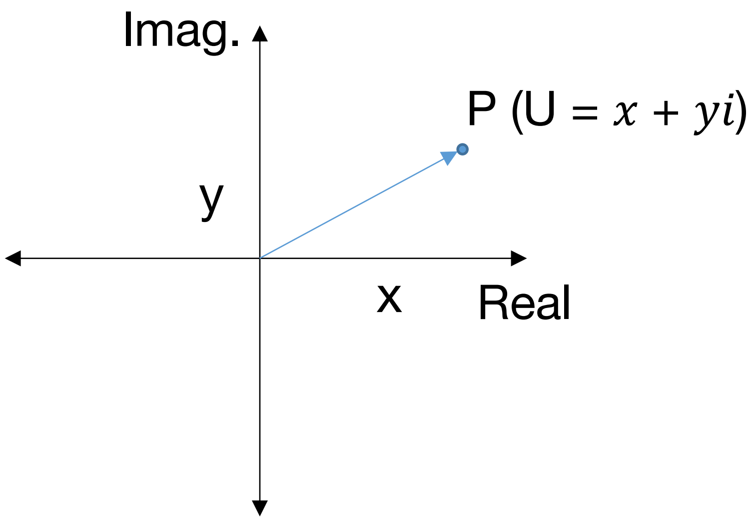
$$A(\mathbf{r}) \cos(\mathbf{kr} - \omega t) \;\text{->}\; A(\mathbf{r}) \cos(\mathbf{kr})$$

**Simplification #2:** We'll use complex numbers when required, it'll make our lives easier. This leads to the *complex* field, U(**r**):

$$A(\mathbf{r}) \cos(\mathbf{kr}) \;\text{<->}\; A(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}} = U(\mathbf{r})$$

## Some things you need to recall about complex numbers

$$U = x + iy,\ i = \sqrt{-1}$$



## Some things you need to recall about complex numbers

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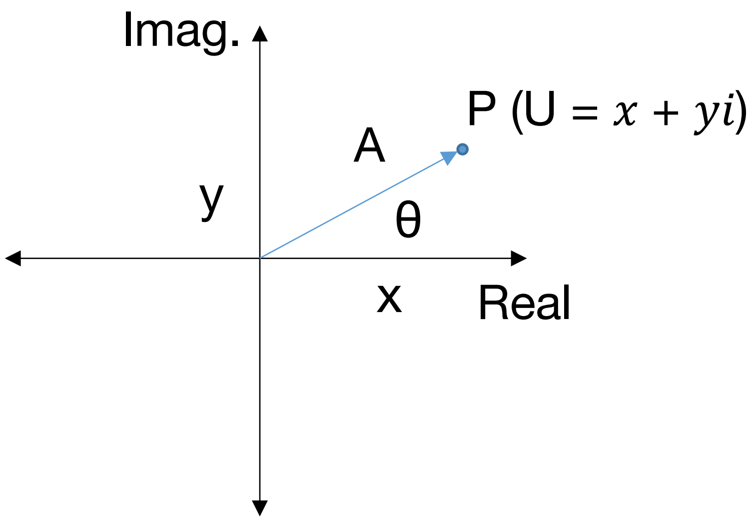


$A = \sqrt{(x^2 + y^2)}$

$\theta = \text{atan}(y/x)$

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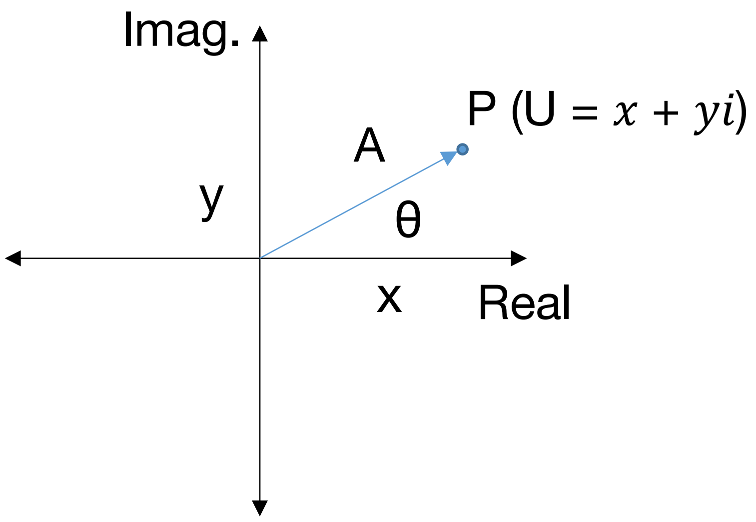


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More useful representation:

$x = A \cos\theta$
$y = A \sin\theta$
$U = A (\cos\theta + i \sin\theta)$

$$\boxed{U = A e^{i\theta}}$$

A = Amplitude of field

θ = Phase of field

## Linear systems and the black box

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$$A(\mathbf{r})\cos(\mathbf{kr}) \leftrightarrow \boxed{A(\mathbf{r})\, e^{i\mathbf{k}\cdot\mathbf{r}} = U(\mathbf{r})}$$

We'll work with complex signals of this form

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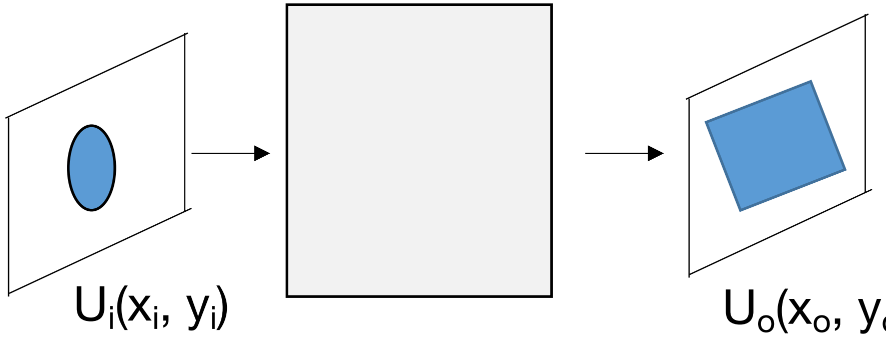
**Simplification #3:** Just consider mappings between planes across space. This is a critically important way of thinking for optics. Think "index card 1 to index card 2".

$$U(\mathbf{r}) \rightarrow U(x,y)$$



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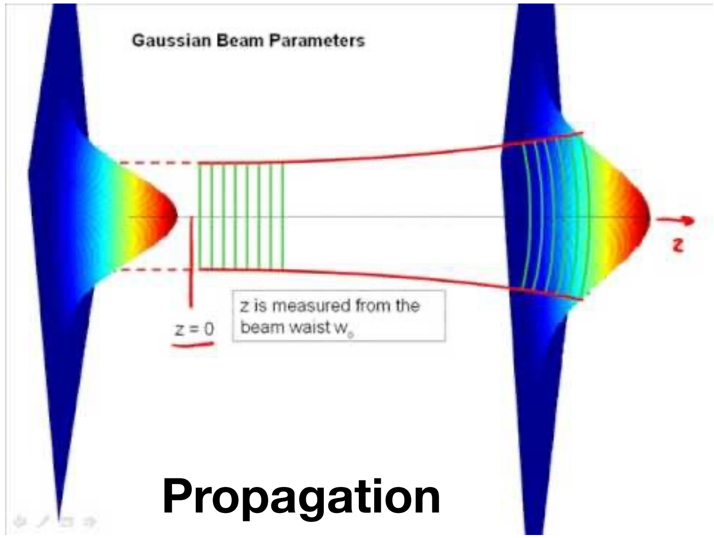
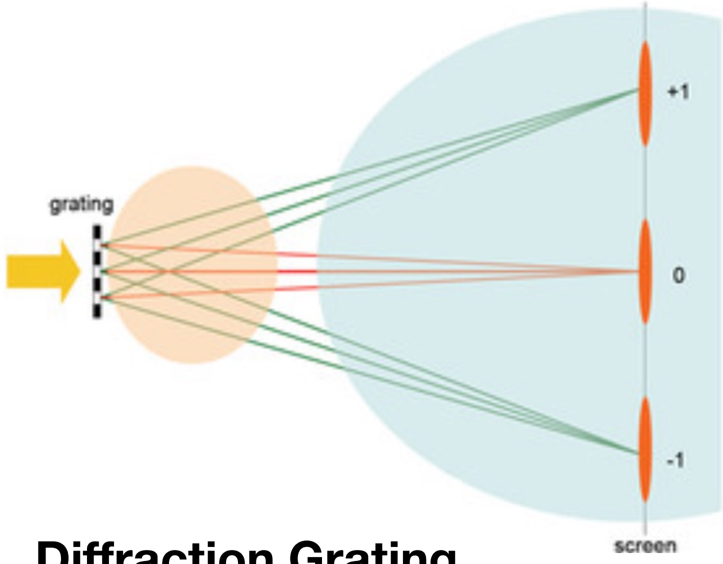
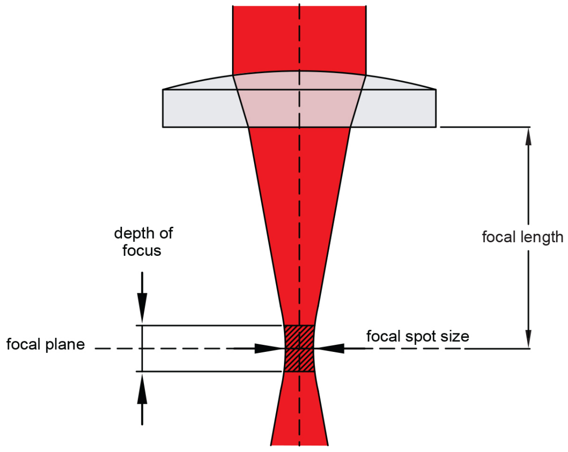
# Linear systems and the black box




Propagation



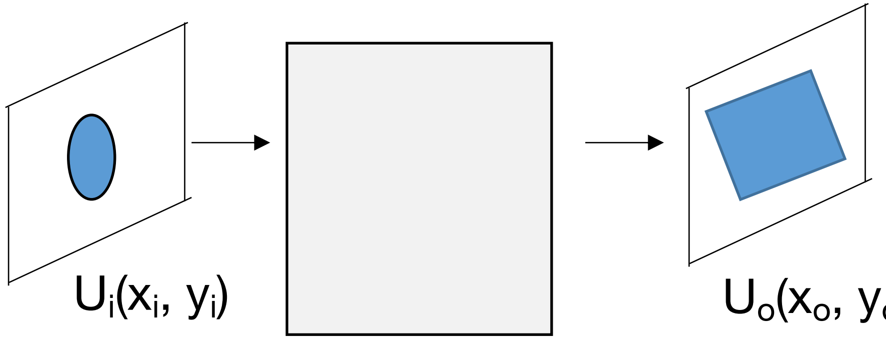

Diffraction Grating




Lens focusing

**Simplification #3:** Just consider mappings between planes across space. This is a critically important way of thinking for optics. Think “index card 1 to index card 2”.

U(**r**) -> U(x,y)



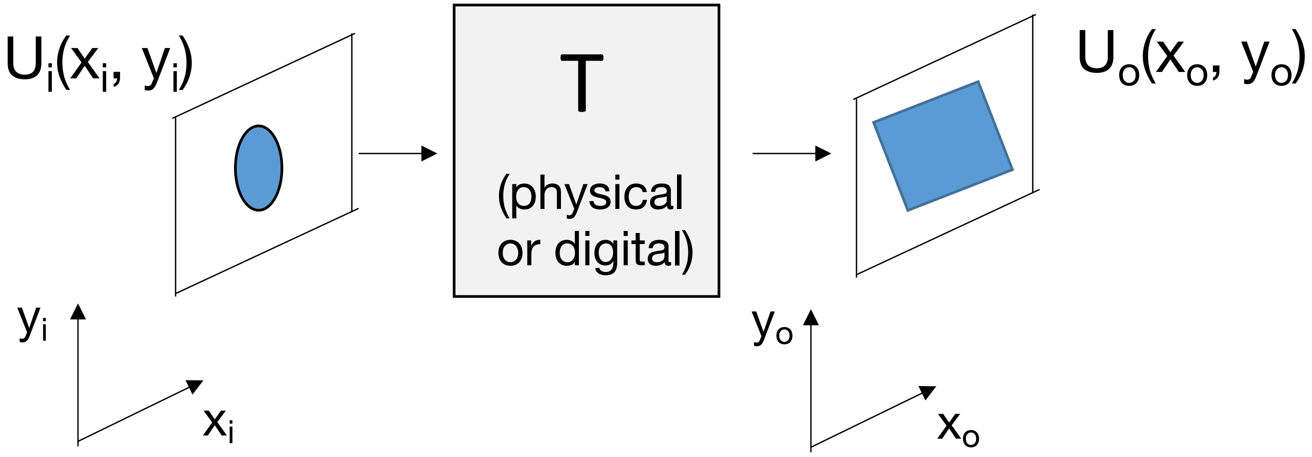
# Linear systems and the black box

**The "optical" black box system:**

An optical black box system maps an input function $U_i(x_i,y_i)$ to an output function $U_o(x_o,y_o)$ via a transform T:

$$U_o(x_o,y_o) = T\,[\,U_i(x_i,y_i)\,]$$

Where T[ ] denotes the optical black box transformation



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Important properties of linear systems:

1. Homogeneity and additivity (superposition):

$$T\,[aU_1(x,y) + bU_2(x,y)] = aT\,[U_1(x,y)] + bT\,[U_2(x,y)]$$

# Linear systems and the black box

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$$T\,[aU_1(x,y) + bU_2(x,y)] = aT\,[U_1(x,y)] + bT\,[U_2(x,y)]$$

2. Shift invariance: for shift distances $d_x$ and $d_y$, we assume that,

$$U_o(x_o - d_x, y_o - d_y) = T\,[U_i(x_i - d_x, y_i - d_y)]$$

# Black box transforms as a convolution



Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

Input Dirac delta function into the black box:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

## Black box transforms as a convolution

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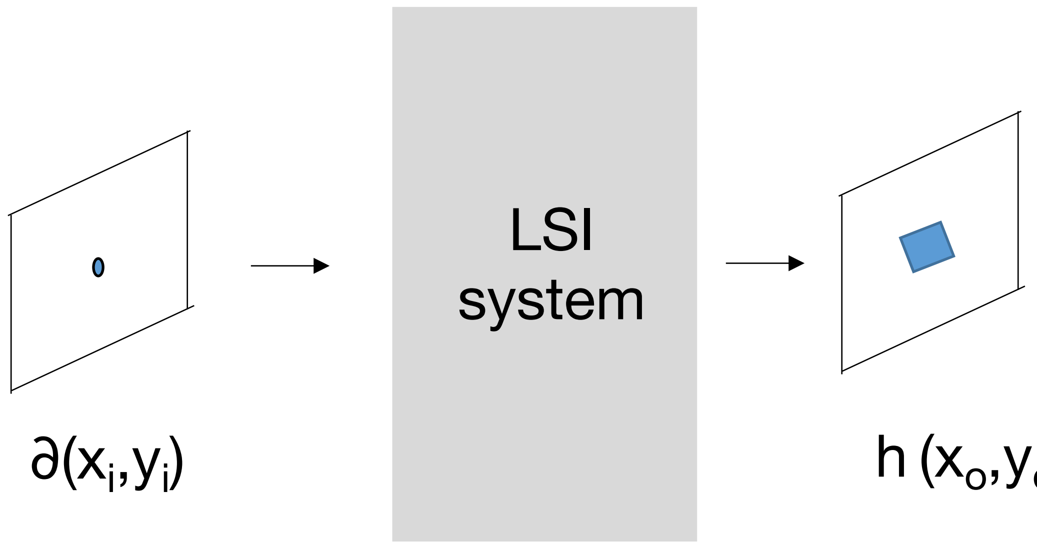
A “perfect” point source



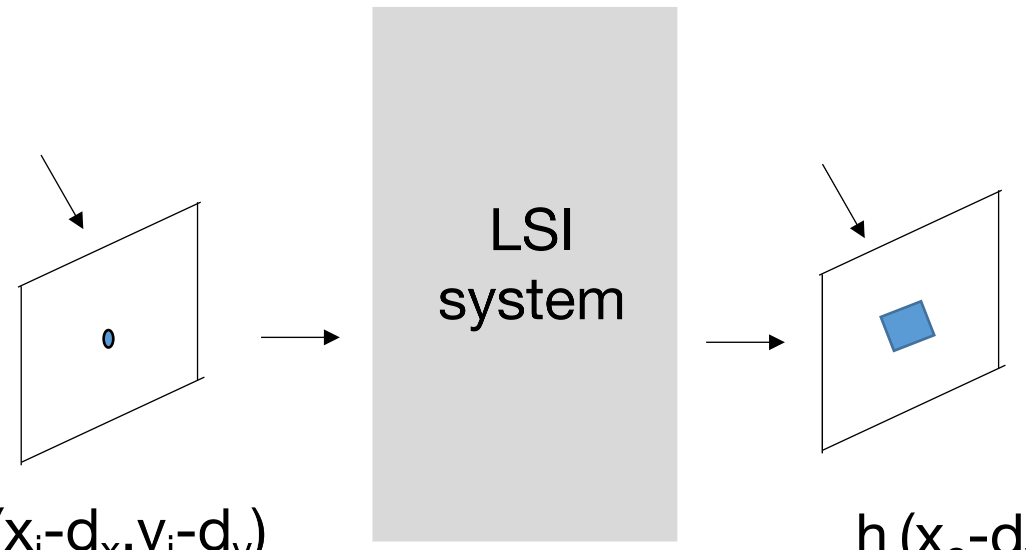

$$h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$$

# Black box transforms as a convolution

Assuming 1) linearity and 2) shift-invariance, we can model any black box with 1 piece of information:

We know the system is shift invariant:

A "perfect" point source




$\partial(x_i-d_x,y_i-d_y)$ $h\,(x_o-d_y,y_o-d_y)$

$$h(x_o-d_y,y_o-d_y) = T\,[\,\partial(x_i-d_x,y_i-d_y)\,]$$

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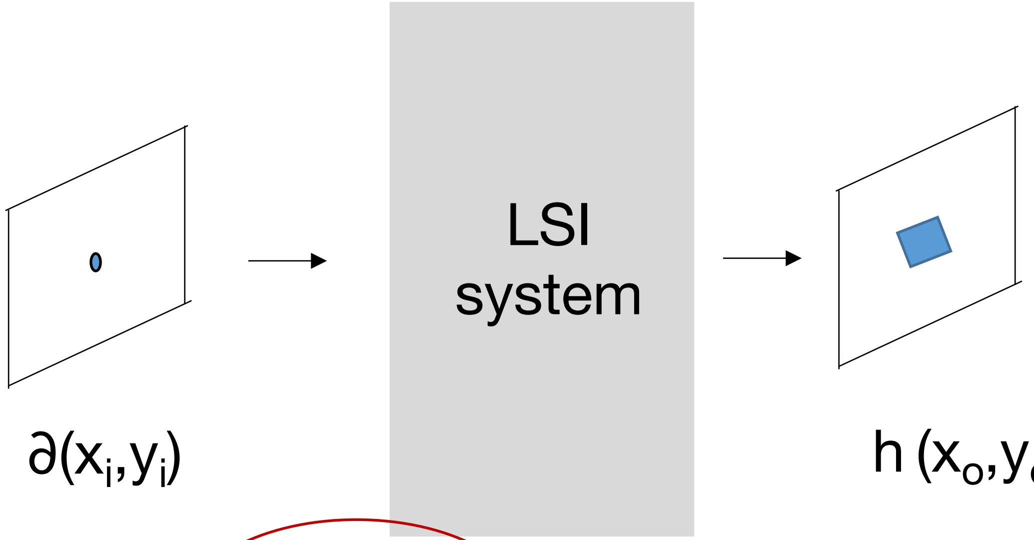
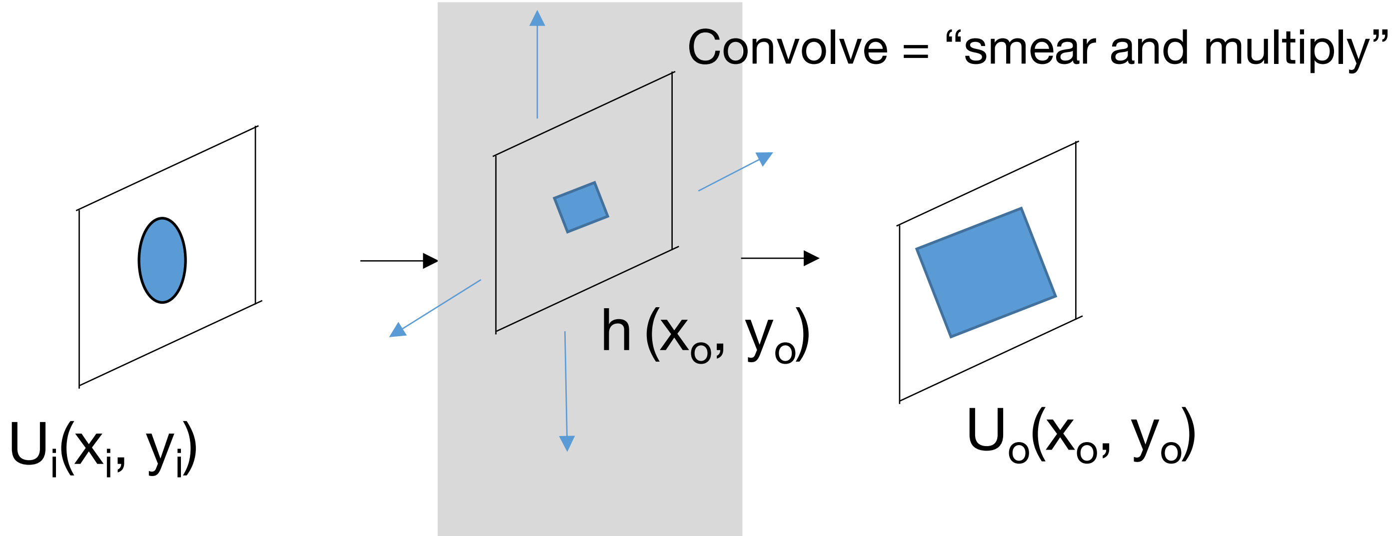



$h(x_o,y_o)$ is the system's point-spread function

Point-spread function $h(x_o,y_o) = T\,[\,\partial(x_i,y_i)\,]$

## Black box transforms as a convolution

Knowing the point-spread function, it is direct to model any output of the black box, given an input:




$$U_o(x_o, y_o) = \iint_{-\infty}^{\infty} U_i(x_i, y_i) h(x_o - x_i, y_o - y_i) dx_i dy_i$$

**Output of linear system is a convolution of the input with its point-spread function**

# Quick proof: The point-spread function forms any output via a convolution in a black-box model

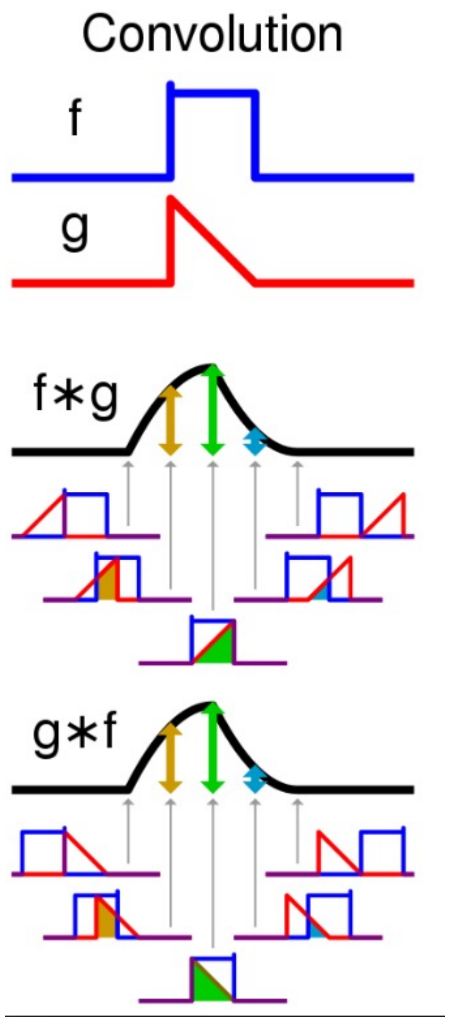
$U_o(x_o,y_o) = T\,[\,U_i(x_i,y_i)\,]$

$U_o(x_o,y_o) = T\,[\int\int U_i(x_i,y_i)\,\partial(x_i\text{-}x_o,\ y_i\text{-}y_o)\,dx_o\,dy_o]$ — Sifting property of delta function

$U_o(x_o,y_o) = \int\int U_i(x_i,y_i)\,T[\partial(x_i\text{-}x_o,\ y_i\text{-}y_o)]\,dx_o\,dy_o$ — Linearity

$U_o(x_o,y_o) = \int\int U_i(x_i,y_i)\,h(x_i\text{-}x_o,\ y_i\text{-}y_o)\,dx_o\,dy_o$ — Shift Invariance

## 1D convolution example




https://en.wikipedia.org/wiki/Convolution

Steps to perform a convolution:

1. Flip one signal (the second one = the PSF)

2. Position PSF right before overlap

With incremental steps:

3. Step PSF over to position $x_o$

4. Compute *area* of overlap of two functions

5. Convolution value at $x_o$= area of overlap

6. Repeat 3-5 until signals do not overlap

# 2D convolution example

- Direct extension of 1D concept to 2D functions
- Note – it is effectively the same with discrete functions = matrices

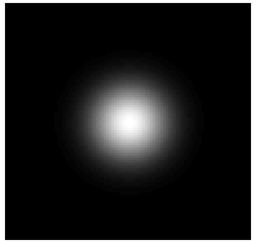
$U_1(x,y)$

y

x

$*$

y2



x2

=

$U_0(x,y)$

y

x

## 2D convolution example

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High-res. real-world object

$U_1(x,y)$

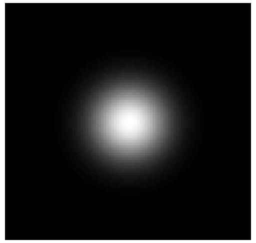
y

x

Blur caused by camera lens

$*$

y2



x2

=

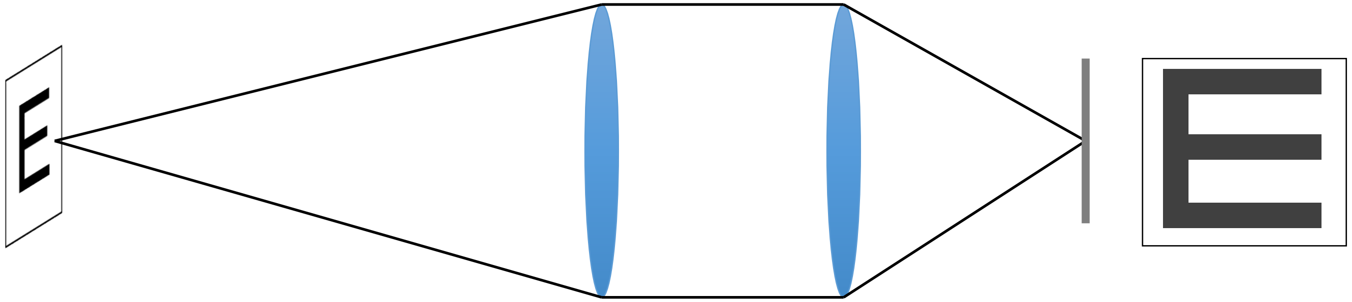
Image at camera sensor plane

$U_0(x,y)$

y

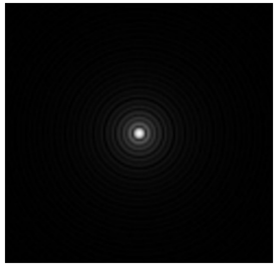
x

# Optical modification Ex. #1: The cubic phase mask



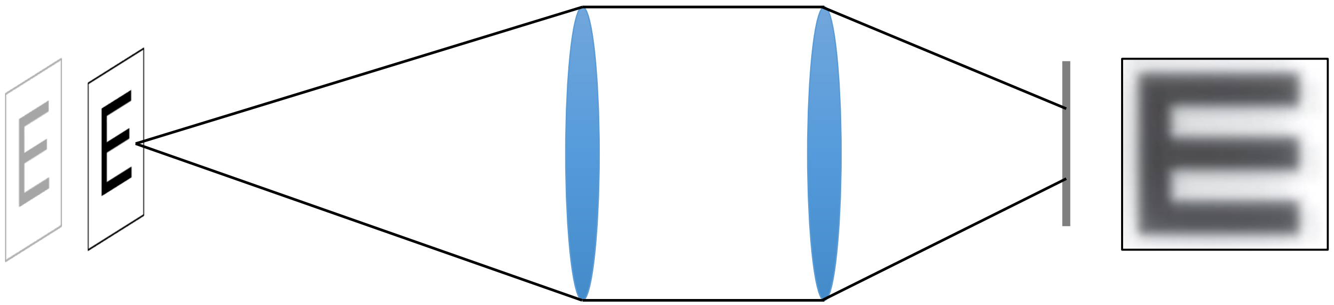
Standard camera
Point-spread function

in focus



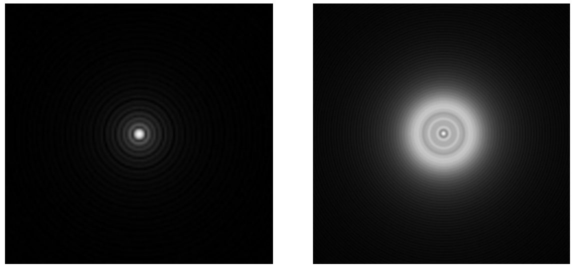
deep imaging

# Optical modification Ex. #1: The cubic phase mask

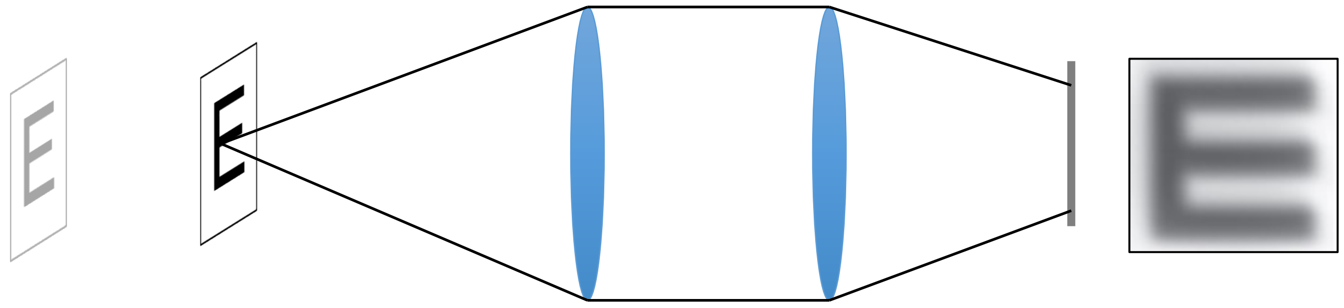


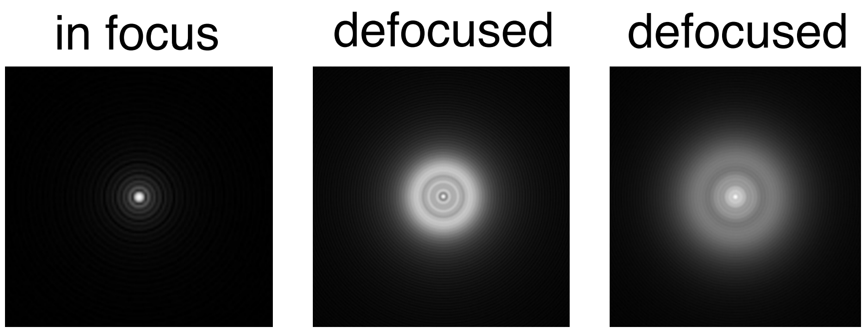
Standard camera:
Limited depth-of-field

in focus defocused



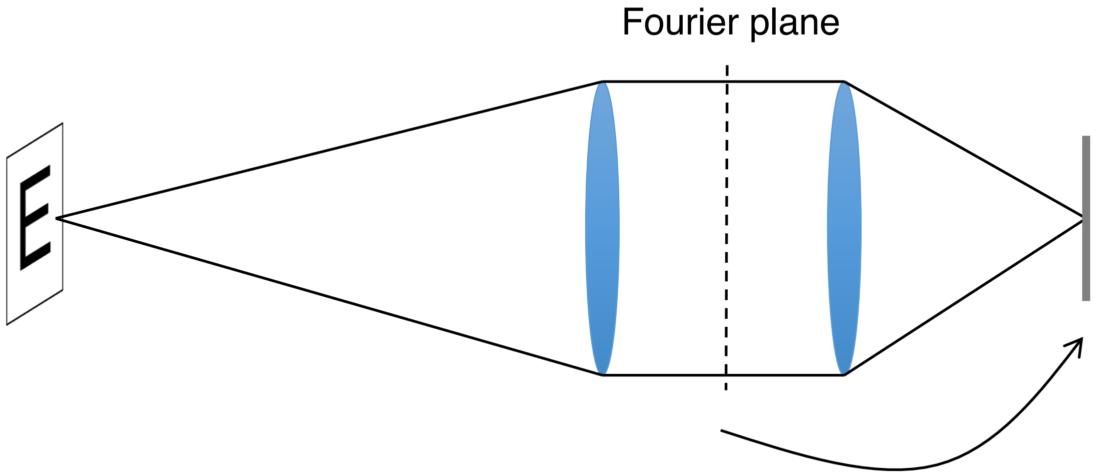
# Optical modification Ex. #1: The cubic phase mask






Standard camera:
Limited depth-of-field

# Optical modification Ex. #1: The cubic phase mask

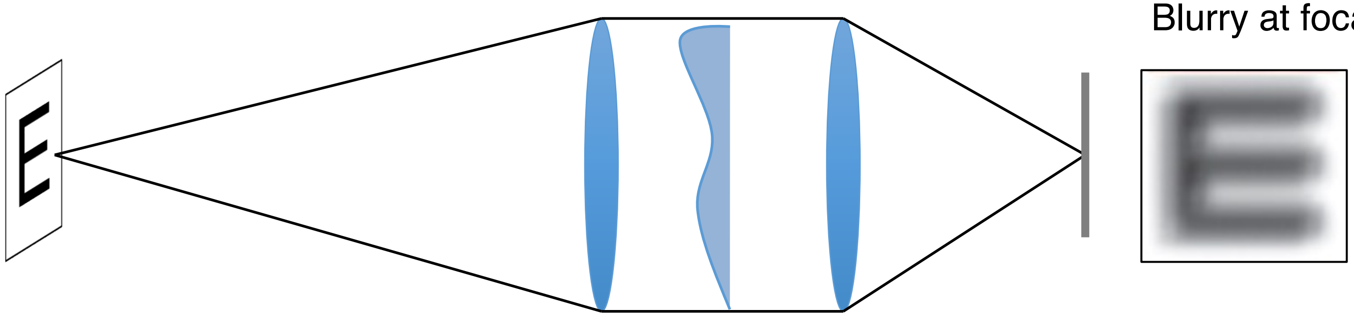



Blur proportional to Fourier transform of shape of aperture

# Optical modification Ex. #1: The cubic phase mask

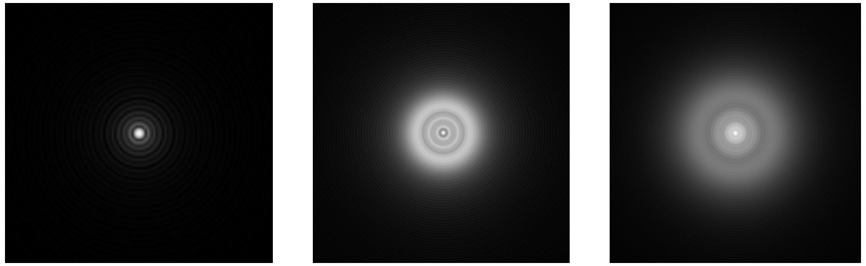
Insert CPM

Blurry at focal plane

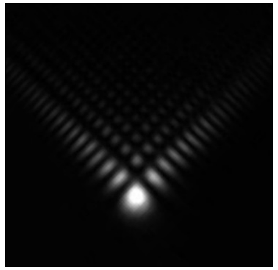


in focus defocused defocused

Standard camera:
Limited depth-of-field



Cubic phase mask:
extended depth-of-field

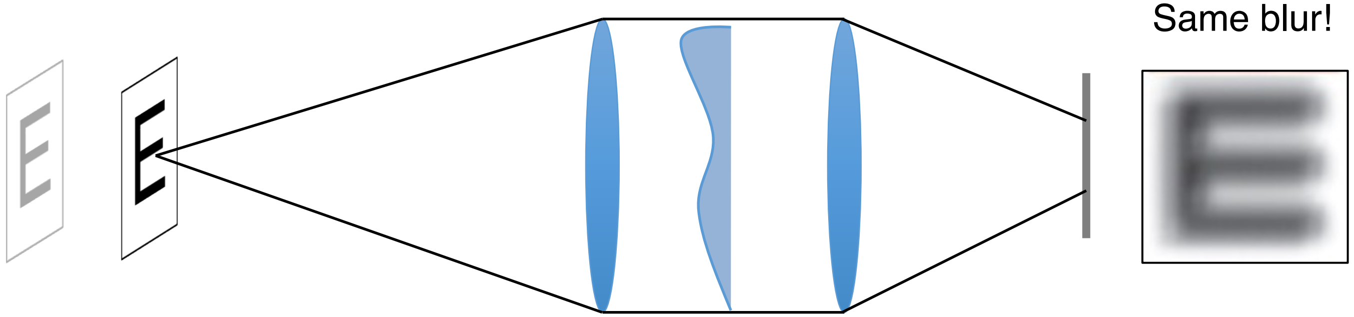


CPM Phase profile



E. Dowski and W. T Cathey, "Extended depth of field through wave-front coding," Appl Opt. 1994

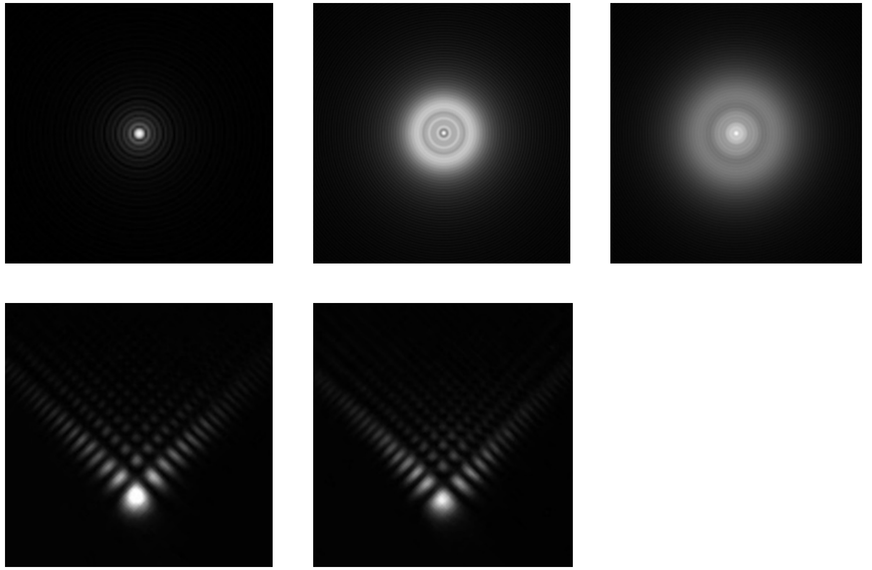
# Optical modification Ex. #1: The cubic phase mask




in focus | defocused | defocused

Standard camera: Limited depth-of-field

Cubic phase mask: extended depth-of-field

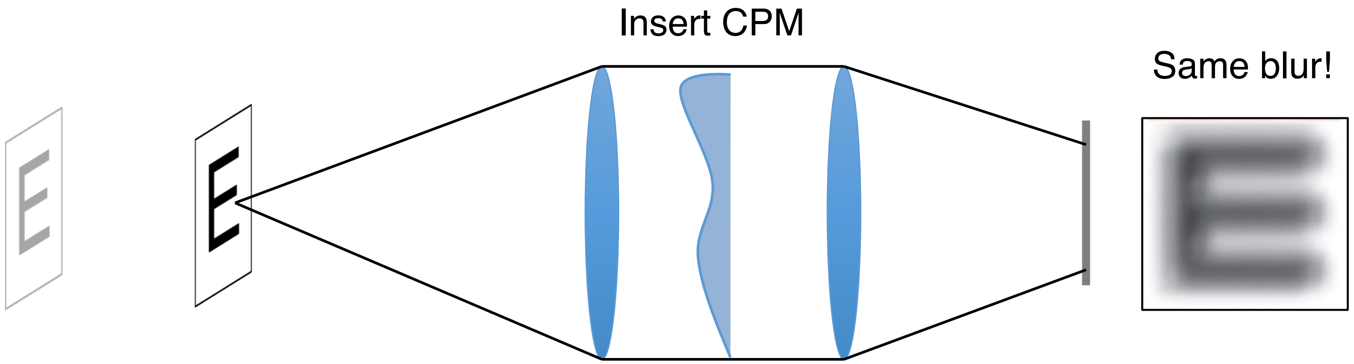


CPM Phase profile



E. Dowski and W. T Cathey, "Extended depth of field through wave-front coding," Appl Opt. 1994

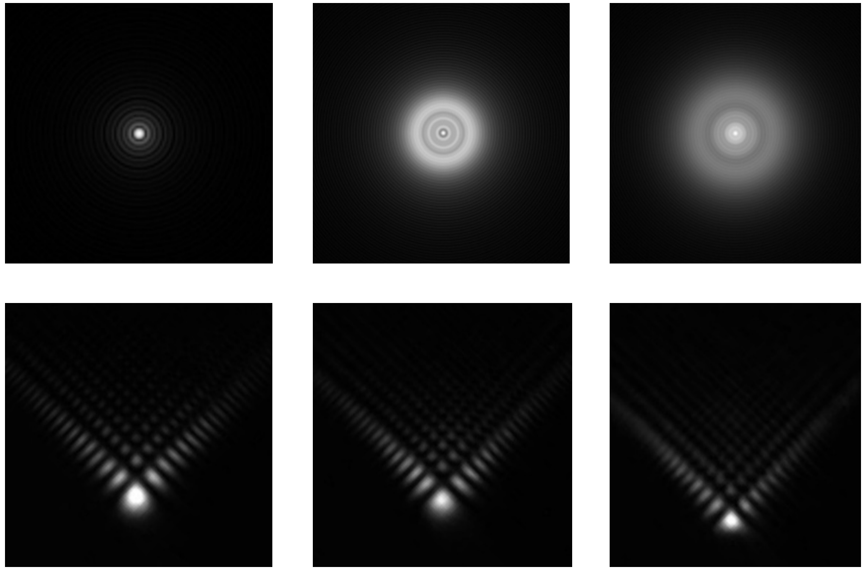
# Optical modification Ex. #1: The cubic phase mask




in focus | defocused | defocused

Standard camera:
Limited depth-of-field

Cubic phase mask:
extended depth-of-field

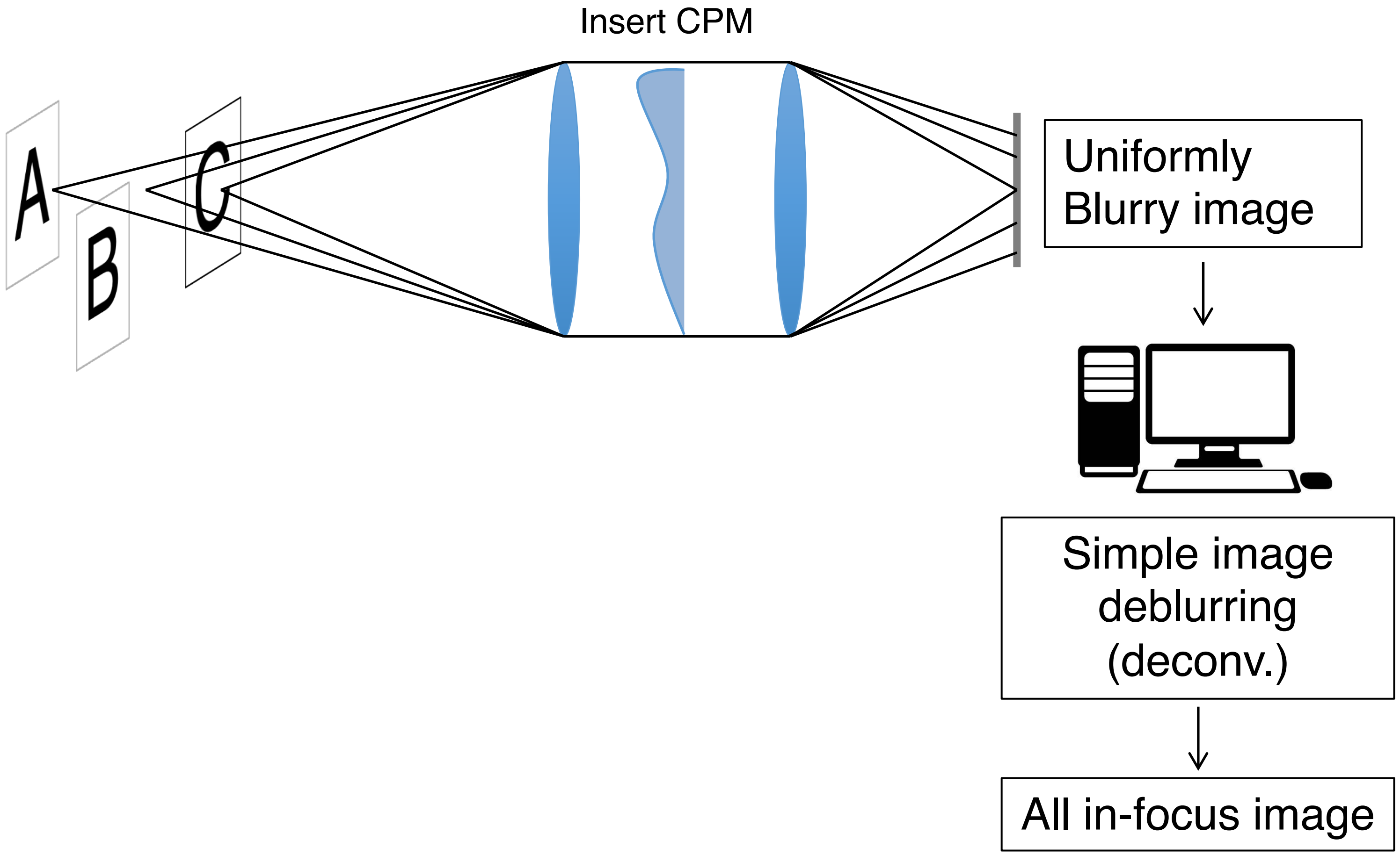


CPM Phase profile



E. Dowski and W. T Cathey, "Extended depth of field through wave-front coding," Appl Opt. 1994

# Optical modification Ex. #1: The cubic phase mask

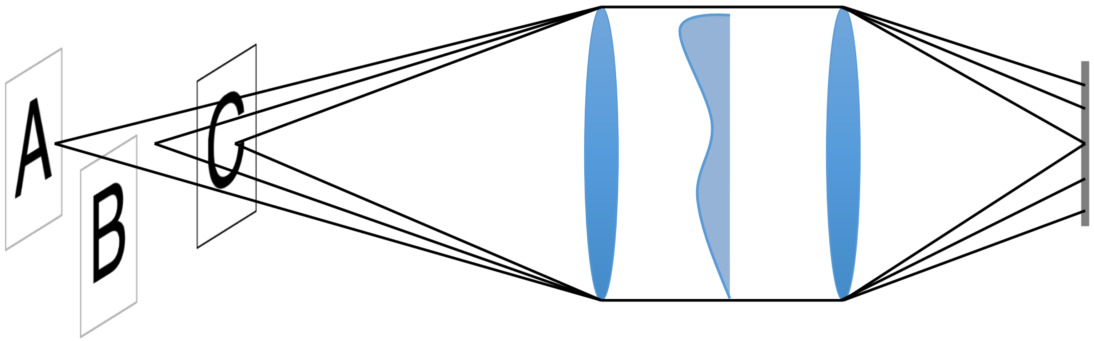





E. Dowski and W. T Cathey, "Extended depth of field through wave-front coding," Appl Opt. 1994

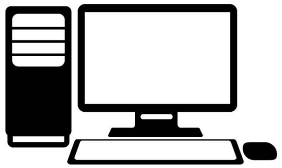
# Optical modification Ex. #1: The cubic phase mask

Insert CPM




Uniformly Blurry image

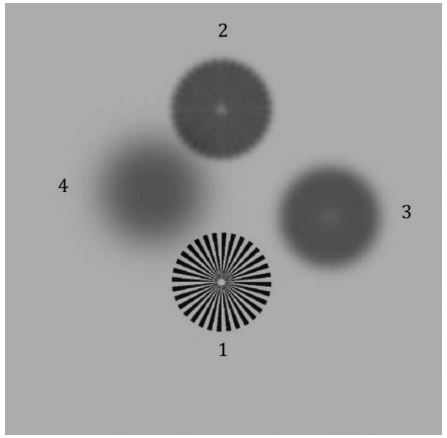
↓



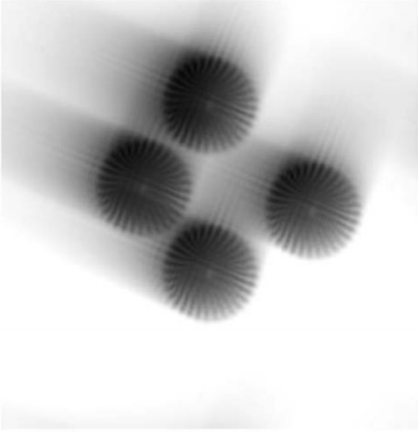
Simple image deblurring (deconv.)

↓

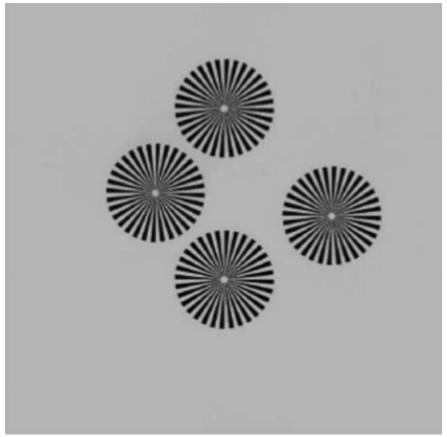
All in-focus image




Standard camera image



CPM image, raw



CPM image, deconvolved

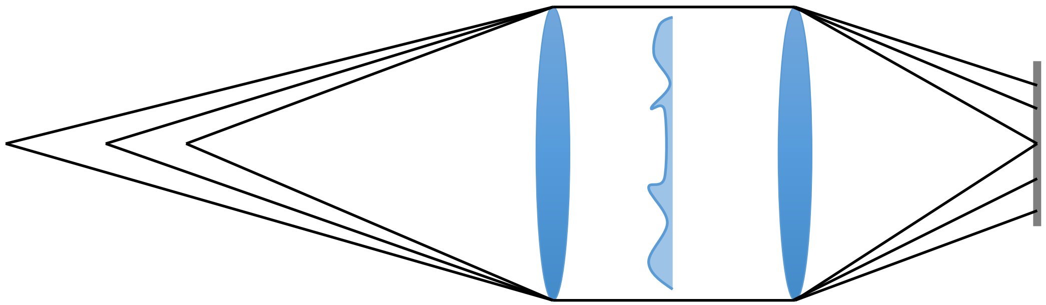


E. Dowski and W. T Cathey, "Extended depth of field through wave-front coding," Appl Opt. 1994

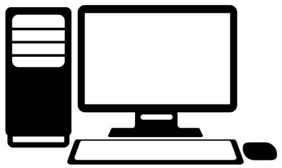
# Optical modification Ex. #1b: Double helix mask

deep imaging

Insert DHM

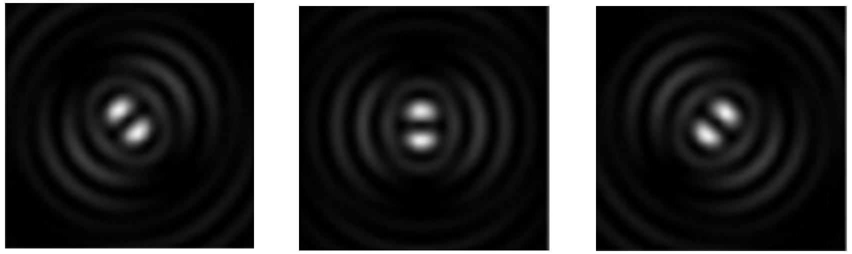


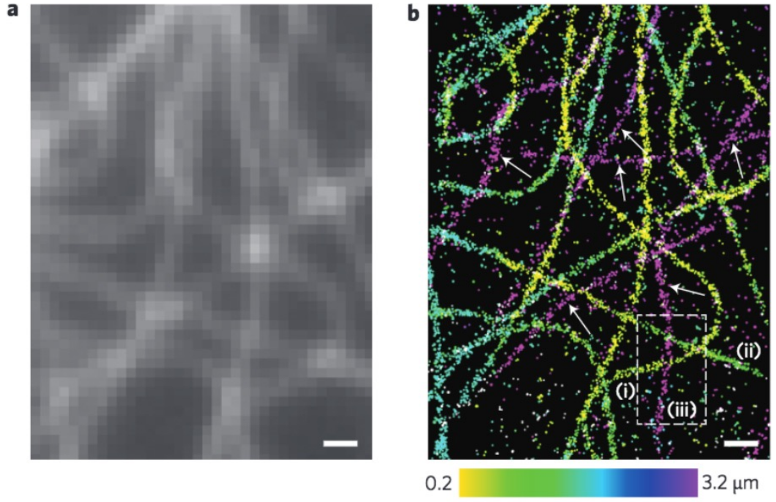
Depth-varying image



Depth detection

in focus | defocused | defocused






Moerner Lab
Nobel Prize in Chemistry, 2014

Jia et al., Nature Photonics 2014

## Useful properties of the convolution

deep imaging

1. Commutativity $$U(x) * h(x) = h(x) * U(x)$$

$\Rightarrow$ You can choose which signal to "flip"

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deep imaging

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$\Rightarrow$ Can change order $\rightarrow$ sometimes one order is easier than another

## Useful properties of the convolution

deep imaging

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$\Rightarrow$ Can change order $\rightarrow$ sometimes one order is easier than another

3. Distributivity $$U(x) * [h_1(x) + h_2(x)] = U(x) * h_1(x) + U(x) * h_2(x)$$

Next : Analyzing light and image formation via Fourier transforms!