Lecture 18: Wave optics and Fourier optics

Machine Learning and Imaging

BME 548L
Roarke Horstmeyer
First - what is light and how can we model it?

- Interpretation #1: Radiation (Incoherent)
  - Model: Rays

- Interpretation #2: Electromagnetic wave (Coherent)
  - Model: Waves

- Interpretation #3: Particle
  - Model: Photons

Real, non-negative
Models absorption and brightness
\[ I_{\text{tot}} = I_1 + I_2 \]

Complex field
Models interference
\[ E_{\text{tot}} = E_1 + E_2 \]
Simple mathematical model of incoherent image formation

Object absorption: $I_0(x,y)$
Illumination pattern: $s(x,y)$
Light exiting object surface: $I_e(x,y) = I_0(x,y) \circ s(x,y)$

- Assume incoherent illumination
- Assume thin 2D object
- Object is real, non-negative map of absorption/reflectivity
Simple mathematical model of image formation

Intensity at image plane $I_p = ?$

$S$

$I_0$

$I_e = S I_0$

Convolution filter $h$

$I_p = H I_e = H S I_0$
Summary: simple physical layers for incoherent imaging

- Deal with sample/image intensities $I$, real and non-negative

- Effect of illumination is element-wise multiplication
  \[ I_{e}(x,y) = S I_{0}(x,y) \]

- Imaging systems blur the object via point-spread function matrix $H$
  \[ I_{b}(x,y) = H I_{0}(x,y) \]

- Discrete pixels down-sample the object via
  \[ I_{d}(x,y) = D I_{0}(x,y) \]

- Add noise into measurement
  \[ I_{N}(x,y) = D I_{0}(x,y) + N \]

- Different colors add linearly
  \[ I_{s}(x, y) = \sum I_{0}(x, y, \lambda) \]
What is light and how can we model it?

• Interpretation #1: Radiation (*Incoherent*)
  • Model: Rays

• Interpretation #2: Electromagnetic wave (*Coherent*)
  • Model: Waves

This class: Modeling coherent radiation as a wave

- Real, non-negative
- Models absorption and brightness
  \[ I_{\text{tot}} = I_1 + I_2 \]

- Complex field
- Models interference
  \[ E_{\text{tot}} = E_1 + E_2 \]
Let’s take a step back: how does light propagate?

Maxwell’s equations without any charge

\[
\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}
\]

\[
\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}
\]

\[
\nabla \cdot \varepsilon \vec{E} = 0
\]

\[
\nabla \cdot \mu \vec{H} = 0.
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Let’s take a step back: how does light propagate?

Maxwell’s equations without any charge:

\[ \nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \]
\[ \nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} \]
\[ \nabla \cdot \varepsilon \vec{E} = 0 \]
\[ \nabla \cdot \mu \vec{H} = 0. \]

1. Take the curl of both sides of first equation
2. Substitute 2\textsuperscript{nd} and 3\textsuperscript{rd} equation
3. Arrive at the wave equation:

\[ \nabla^2 \vec{E} - \frac{n^2}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]
\[ n = \left( \frac{\varepsilon}{\varepsilon_0} \right)^{1/2} \]
\[ c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}. \]
Let’s take a step back: how does light propagate?

Considering light that isn’t pulsed over time, we can use the following solution:

\[ u(P, t) = A(P) \cos[2\pi vt + \phi(P)] \]

\[ u(P, t) = \text{Re}\{U(P) \exp(-j2\pi vt)\}, \]
Let’s take a step back: how does light propagate?

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With this particular solution, we get the following important time-independent equation:

\[ (\nabla^2 + k^2)U = 0. \]

\[ k = \frac{2\pi n v}{c} = \frac{2\pi}{\lambda}. \]
Let’s take a step back: how does light propagate?

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\[ (\nabla^2 + k^2)U = 0. \]

\[
\begin{align*}
    k &= 2\pi n \frac{\nu}{c} = \frac{2\pi}{\lambda}, \\

    U(P_2) &= \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{21})}{r_{21}} \cos \theta \, ds
\end{align*}
\]

This is an important equation in physics. We won’t go into the details, but it leads to the Huygen-Fresnel principle:
Plane-to-plane light propagation via the "paraxial approximation"

The Huygens-Fresnel Equation

\[ U(P_2) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{21})}{r_{21}} \cos \theta \, ds \]
Plane-to-plane light propagation via the "paraxial approximation"

The Huygens-Fresnel Equation

\[ U(P_2) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{21})}{r_{21}} \cos \theta \, ds \]

Generally connects two points in 3D:

\[ U(P_1) = U(x_1, y_1, z_1) \]

\[ U(P_2) = U(x_2, y_2, z_2) \]
Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):

\[ U(P_1) = U(x_1, y_1, z_1 = z_{p1}) \quad U(P_2) = U(x_2, y_2, z_2 = z_{p2}) \]

\[ U(P) = E(x, y, z)e^{ikz} \]
Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):

Paraxial approximation:

\[
\nabla^2 U + 2ik \frac{dU}{dz} = 0
\]

\[
\nabla^2 \text{def} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

\[
U(P_1) = U(x_1, y_1, z_1 = z_{p1}) \quad U(P_2) = U(x_2, y_2, z_2 = z_{p2})
\]

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U(P) = E(x, y, z)e^{ikz}
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Plane-to-plane light propagation via the “paraxial approximation”

We are usually concerned about propagation between two planes (almost always in an optical system):

**Paraxial approximation:**

\[ \nabla_{\perp}^2 U + 2ik \frac{dU}{dz} = 0 \]

Substitute in \( U(P) = E(x, y, z)e^{ikz} \) and crank the wheel,

\[ \nabla_{\perp}^2 E + 2ik \frac{dE}{dz} + 2k^2 E = 0 \]

Paraxial Helmholtz Equation. This has an exact integral solution:
Plane-to-plane light propagation via the “paraxial approximation”

We are usually concerned about propagation between two planes (almost always in an optical system):

**Paraxial approximation:**

\[
\nabla^2_1 U + 2ik \frac{dU}{dz} = 0 \quad \text{Substitute in } \ U(P) = E(x, y, z)e^{ikz} \text{ and crank the wheel,}
\]

\[
\nabla^2_1 E + 2ik \frac{dE}{dz} + 2k^2 E = 0 \quad \text{Paraxial Helmholtz Equation. This has an exact integral solution:}
\]

\[
E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \int \int_{-\infty}^{+\infty} E(x', y', 0)e^{\frac{ik}{2z}[(x-x')^2+(y-y')^2]} \, dx' \, dy'
\]

This is how light propagates from one plane to the next. It’s a convolution!
Fresnel light propagation as a convolution

\[ E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z} \left[(x-x')^2 + (y-y')^2\right]} \, dx' \, dy' \]

\[ h(x, y, z) = \frac{e^{ikz}}{i\lambda z} e^{i \frac{k}{2z} (x^2 + y^2)} \]

\[ E(x, y, z) = E(x, y, 0) * h(x, y, z) \]
Fresnel light propagation as a convolution

\[ E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z}[(x-x')^2+(y-y')^2]} \, dx' \, dy' \]

\[ h(x, y, z) = \frac{e^{ikz}}{i\lambda z} e^{\frac{k}{2z}(x^2+y^2)} \]

\[ E(x, y, z) = E(x, y, 0) * h(x, y, z) \]
From the Fresnel approximation to the Fraunhofer approximation

Fresnel Approximation:

\[ E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0)e^{\frac{ik}{2z}[(x-x')^2+(y-y')^2]} \, dx' \, dy' \]

Let's assume that the second plane is "pretty far away" from the first plane. Then,

\[ z > \frac{2D^2}{\lambda} \]
From the Fresnel approximation to the Fraunhofer approximation

**Fresnel Approximation:**

\[
E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \int \int_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z} [(x-x')^2 + (y-y')^2]} \, dx' \, dy'
\]

Let's assume that the second plane is "pretty far away" from the first plane. Then,\[z > \frac{2D^2}{\lambda}\]

1. Expand the squaring

\[
E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \int \int E(x', y', 0) e^{\frac{ik}{2z} (x^2 + y^2)} e^{\frac{ik}{2z} (x'^2 + y'^2)} e^{\frac{ik}{2z} (xx' + yy')} \, dx' \, dy'
\]
From the Fresnel approximation to the Fraunhofer approximation

**Fresnel Approximation:**

\[ E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z}[(x-x')^2+(y-y')^2]} \, dx' \, dy' \]

Let's assume that the second plane is “pretty far away” from the first plane. Then,

\[ z > \frac{2D^2}{\lambda} \]

1. Expand the squaring

\[ E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint E(x', y', 0) e^{\frac{ik}{2z}(x^2+y^2)} e^{\frac{ik}{2z}(x'^2+y'^2)} e^{\frac{ik}{2z}(xx'+yy')} \, dx' \, dy' \]

2. Front term comes out, assume second term goes away, then,

\[ E(x, y, z) = C \iint E(x', y', 0) e^{\frac{ik}{2z}(xx'+yy')} \, dx' \, dy' \]

\[ C = \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{2z}(x^2+y^2)} \]

**Fraunhofer diffraction is a Fourier transform!!!!!!!**
This is the aperture

This is what you see far away

Two-dimensional rectangle function as an image
d) Magnitude of Fourier spectrum of the 2-D rectangle
Machine Learning and Imaging – Roarke Horstmeyer (2022)

- **Deep Imaging**

- **Magnitude of Cheetah**

- **Phase of Cheetah**

- **Magnitude of Zebra**

- **Phase of Zebra**
Model of a microscope (or camera) using Fourier transforms:

\[ E_s(x_s, y_s, 0) \quad \rightarrow \quad E_a(x_a, y_a, w) \]

2D Fourier Transform
Model of a microscope (or camera) using Fourier transforms:

Effect of the lens is to block light.

Use *thin object approximation* to determine distribution of light on the immediate other side of the lens stop:

\[ E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d) \]

\[ E_s(x_s, y_s, 0) \rightarrow E_a(x_a, y_a, w) \]

2D Fourier Transform

1's in here

0's out here
Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?

\[ E_s(x_s, y_s, 0) \xrightarrow{2D \text{ Fourier Transform}} E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d) \]
Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?

inverse Fourier transform!

\[ E_s(x_s, y_s, 0) \quad \xrightarrow{\text{2D Fourier Transform}} \quad E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d) \quad \xrightarrow{\text{2D inverse Fourier Transform}} \]
This process should sound familiar…

\[
U_1(x,y) \ast h = U_2(x,y)
\]

\[
\hat{U}_1(f_x, f_y) \cdot \hat{H} \Rightarrow \hat{U}_2(f_x, f_y)
\]
Model of image formation for wave optics (coherent light):

1. Discrete sample function $s(x,y)$ (complex)
Model of image formation for wave optics (coherent light):

1. Discrete sample function $s(x,y)$ (complex)
2. Compute its 2D Fourier transform $\hat{s}(f_x, f_y)$

“Fourier plane”
Model of image formation for wave optics (coherent light):

1. Discrete sample function \( s(x,y) \) (complex)
2. Compute its 2D Fourier transform \( \hat{s}(f_x, f_y) \)
3. Multiply by “aperture” function \( A(f_x, f_y) \)

2D FT

“Fourier plane”
Model of image formation for wave optics (coherent light):

1. Discrete sample function $s(x, y)$ (complex)

2. Compute its 2D Fourier transform $\hat{s}(f_x, f_y)$

3. Multiply by “aperture” function $A(f_x, f_y)$

4. Compute inverse Fourier transform $s'(x', y')$ (complex)
Model of image formation for wave optics (coherent light):

1. Discrete sample function \( s(x,y) \) (complex)
2. Compute its 2D Fourier transform \( \hat{s}(f_x, f_y) \)
3. Multiply by “aperture” function \( A(f_x, f_y) \)
4. Compute inverse Fourier transform \( s'(x,y') \) (complex)
5. Detector measures \( |s'(x',y')|^2 \)

2D FT  \quad \text{“Fourier plane”}  \quad \text{2D IFT}  \quad \text{“Blurred image”}
Can also model this using the Convolution Theorem

Aperture function (lens shape) \[ A(f_x, f_y) \]

Camera blur function (IFT of lens shape) \[ h(x, y) \]

2D IFT
Two modeling choices for the camera:

**Spatial Frequency Domain**

\[
\hat{S}_1(f_{xi}, f_{yi}) \rightarrow A(f_x, f_y) \rightarrow \hat{S}_2(f_{xo}, f_{yo})
\]

**Spatial Domain**

\[
s_1(x_i, y_i) \rightarrow h(x, y) \rightarrow \text{Convolve = blur} \rightarrow s_2(x_o, y_o)
\]
Linear systems and the black box

The optical black box system and the point-spread function:

Light $g_1(x_i, y_i)$ entering “black box” optical system modified by system point-spread function $h(x_2, y_2)$

$$g_2(x_2, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(x_1, y_1) h(x_2 - x_1, y_2 - y_1) \, dx_1 \, dy_1$$

Assume shift invariance:

This is the system point-spread function
Summary of two models for image formation

- **Interpretation #1: Radiation (Incoherent)**
  - Model: Rays

  \[ I_{\text{tot}} = I_1 + I_2 \]

- Real, non-negative
- Models absorption and brightness
Mathematical model of for incoherent image formation

- All quantities are real, and non-negative

Object absorption:

Illumination brightness:

$$S_0(x, y)$$

$$B(x, y)$$
Mathematical model of for incoherent image formation

- All quantities are real, and non-negative

Illumination brightness: $S_0(x,y)$

Object absorption: $B(x,y)$

100 photons

60% transmission

60 photons
Mathematical model of for incoherent image formation

- All quantities are real, and non-negative

Illumination brightness: $S_0(x,y)$

Object absorption: $B(x,y)$

- 80 photons
- 8 photons
- 10% transmission
Mathematical model of for incoherent image formation

- All quantities are real, and non-negative

Object absorption:

\[ S_0(x,y) \]

Illumination brightness:

\[ B(x,y) \]

\[ B \cdot S_0 \]

multiplication
Summary of two models for image formation

**Interpretation #1: Radiation (Incoherent)**
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  \[ I_s = B S_0 \]

**Interpretation #2: Electromagnetic wave (Coherent)**
- Model: Waves
- Complex field
- Models Interference
  \[ E_{\text{tot}} = E_1 + E_2 \]
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = $S(x,y)$

(Laser light or Ultrasound)

Incident field: $C(x,y) = A_i(x,y)$

Transmitted field: $U(x,y) = A_t(x,y)$

Point #1: Amplitudes behave just like before

$A_t(x,y) = A_i(x,y) \cdot S(x,y)$

- 100 photons
- 60% transmission
- 60 photons
Mathematical model of for coherent image formation

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Point #1: Amplitudes behave just like before

$Laser light$ or $Ultrasound$

$A_t(x,y) = A_i(x,y) \cdot S(x,y)$

100 photons

20 photons

20% transmission
Mathematical model of for coherent image formation

• Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = S(x,y)

(Laser light or Ultrasound)

Incident field:

\[ C(x,y) = A_i(x,y) \]

Transmitted field:

\[ U(x,y) = A_t(x,y) = A_i(x,y) S(x,y) \]

Point #1: Amplitudes behave just like before

\[ A_t(x,y) = A_i(x,y) S(x,y) \]
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = $S(x,y)$

Sample phase delay = $\exp[ik\phi(x,y)]$

New: complex phase delay
- Needed to represent wave
- Represents wave delay across space

Incident field:

$C(x,y) = A_i(x,y) \exp[ik\phi_i(x,y)]$

Transmitted field:

$U(x,y) = A_i(x,y) S(x,y) \exp[ik\phi_t(x,y)]$
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = $S(x,y)$

Sample phase delay = $\exp[ik\phi(x,y)]$

(Laser light or Ultrasound)

New: complex phase delay

$\phi_i(x,y) = 0$

$\pi/6$ deg. Phase delay
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase delay.

Sample absorption $= S(x,y)$

Sample phase delay $= \exp[ik\phi(x,y)]$

New: complex phase delay

(Laser light or Ultrasound)

$\phi_i(x,y) = 0$

$\pi/6$ deg. Phase delay

Incident light

Transmitted light

Total lag $= \pi/6$ rad
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = S(x,y)

Sample phase delay = \( \exp[\text{i}k\phi(x,y)] \)

(Laser light or Ultrasound)

New: complex phase delay

\( \phi_i(x,y) = \frac{\pi}{8} \text{ rad.} \)

Incident light: \( \pi/8 \) behind

Transmitted light
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = $S(x,y)$

Sample phase delay = $\exp[ik\phi(x,y)]$

(Laser light or Ultrasound)

New: complex phase delay

$\phi(x,y) = \pi/8$ rad.

$
\pi/4$ rad

Phase delay

Incident light: $\pi/8$ behind

Transmitted light

Total lag = $\pi/8 + \pi/4 = 3\pi/8$ rad
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = $S(x,y)$

Sample phase delay = $\exp[ik\phi(x,y)]$

Output phase is sum of phase delays, product of phasors

$\phi_t(x,y) = \phi(x,y) + \phi_i(x,y)$

$\exp[ik\phi_t(x,y)] = \exp[ik\phi_i(x,y)] \exp[ik\phi(x,y)]$

Incident field:

$C(x,y) = A_i(x,y) \exp[ik\phi_i(x,y)]$

Transmitted field:

$U(x,y) = A_i(x,y) S(x,y) \exp[ik\phi_i(x,y)] \exp[ik\phi(x,y)]$
Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption = \( S(x,y) \)

Sample phase delay = \( \exp[i k \phi(x,y)] \)

Incident field: \( C(x,y) = A_i(x,y) \exp[i k \phi_i(x,y)] \)

Transmitted field: 

\[
U(x,y) = A_i(x,y) S(x,y) \exp[i k \phi_i(x,y)] \\
\exp[i k \phi(x,y)]
\]

Conclusion:

Transmitted field = incident field \( \times \) complex sample:

\( U(x,y) = C(x,y) S(x,y) \exp[i k \phi(x,y)] \)
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- **Interpretation #2: Electromagnetic wave (Coherent)**
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  - Models Interference
    \[ E_{\text{tot}} = E_1 + E_2 \]
    \[ U = C S_0 \]
    U, C and S are complex!
Additional Information about sample index of refraction, spatial frequency and Fourier optics
Microscope illumination and sample index of refraction

So far: illuminate the sample and create a field that is equivalent to the sample’s absorption and phase – why is this true???
Microscope illumination and sample index of refraction

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Sample index of refraction $n(x,y,z) = 1 + ia(x) + \phi(x)$

- $a(x=1) = .2$
- $\phi(x=1) = .1$

*For more information, see D. Paganin, Coherent X-Ray Optics, Section 2.2*
Microscope illumination and sample index of refraction

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Thin sample approximation:

Sample’s effect on light is multiplication with $\exp[-ik \ast n(x,y)]$

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In 1D: Emerging field \( U(x) = \text{incident field } U_i(x) \cdot \text{sample function } s(x) \)
Microscope illumination and sample index of refraction

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\[
\begin{align*}
a(x=1) &= .2 \\
\varphi(x=1) &= .1
\end{align*}
\]

Thin sample approximation:

Sample’s effect on light is multiplication with \( \exp[-ik \ast n(x,y)] \)

In 1D: Emerging field \( U(x) = \text{incident field } U_i(x) \ast \text{sample function } s(x)=\exp[-ik \ast n(x)] \)

\[
\begin{align*}
U(x) &= U_i(x) \ast \exp[-ik \ast n(x)] = U_i(x) A(x) \exp[ik\varphi(x)] \\
A(x) &= \exp[k a(x)]
\end{align*}
\]
Microscope illumination and sample index of refraction

Sample absorption = \( A(x) \)
Sample phase = \( \exp[ik\varphi(x)] \)

Emerging field \( U = \text{incident field } U_i(x) \times \text{sample function } s(x) \)
Q: When is the emerging field equal to the absorption and phase?

Sample absorption = $A(x)$
Sample phase = $\exp[\imath k \varphi(x)]$

Emerging field $U =$ incident field $U_i(x) \ast$ sample function $s(x)$
**Microscope illumination and sample index of refraction**

**Q:** When is the emerging field equal to the absorption and phase?

Sample absorption = $A(x)$  
Sample phase = $\exp[\text{ik}\varphi(x)]$

**A:** When the incident wave = 1, means uniform in amplitude and phase:

$U_i(x) = 1 \quad \rightarrow \quad U(x) = A(x) \exp[\text{ik}\varphi(x)]$

Emerging field $U = \text{incident field } U_i(x) \ast \text{sample function } s(x)$
Microscope illumination and sample index of refraction

Q: When is the emerging field equal to the absorption and phase?

Sample absorption = \( A(x) \)
Sample phase = \( \exp[i k \varphi(x)] \)

A: When the incident wave = 1, means uniform in amplitude and phase:

\[
U_i(x) = 1 \quad \rightarrow \quad U(x) = A(x) \exp[i k \varphi(x)]
\]

Plane wave \( U_i(x) = 1 \ast \exp(i k \bullet x) \)

\[
U_i(x) = \exp(i k x \sin(\theta))
\]

\( \theta = 0 \) everywhere

\( \theta = 30 \) everywhere

This is when incident wave hits the sample with \( \theta = 0 \)!
Model of image formation for wave optics (coherent light):

1. Discrete sample function $s(x,y)$ (complex)

2. Compute its 2D Fourier transform $\hat{s}(f_x, f_y)$

What does $f_x$ represent, really?

“Blurred image”
From before: Spatial frequencies = “stripes” within each image

\[
U(x,y) 
\rightarrow \hat{U}(f_x, f_y)
\]

\[
\hat{U}(f_x, f_y) 
\rightarrow U(x,y)
\]

\[f_x = \frac{2\pi}{T_x}\]

\[f_y = \frac{2\pi}{T_y}\]
Ray angle and spatial frequency

Plane of interest

Incident plane wave

Stripes are for complex fields!

Proportional to distance between subsequent peaks of wave along plane of interest
Ray angle and spatial frequency

Distance to two crests = spatial period

\[ \sin(\theta) = \frac{\lambda}{d} \]

\[ d = \frac{\lambda}{\sin(\theta)} \]
Ray angle and spatial frequency

Plane of interest

Distance to two crests = spatial period

\[ \sin(\theta) = \frac{\lambda}{d} \]

\[ d = \frac{\lambda}{\sin(\theta)} \]

Spatial frequency = 1/spatial period
(number of periods per unit length)

\[ f_x = \frac{1}{d} = \frac{\sin(\theta)}{\lambda} \]
Equivalent coordinates in the Fourier domain and at the Fourier plane

\[ f_x = \frac{\sin(\theta)}{\lambda} \]

\[ k_x = 2\pi f_x = \frac{2\pi}{\lambda} \sin(\theta) \]
General rules for applying the Fourier transform in optics

Situation 1: From an object to a plane “really far away”

\[ U_1(x_1) ~ \sim \mathcal{F}[U_1(x_1/M_1)] \]
General rules for applying the Fourier transform in optics

Situation 1: From an object to a plane “really far away”

\[ U_2(x_2) \sim F[U_1(x_1/M_1)] \]

Situation 2: From an object to the back focal plane of the microscope objective lens

\[ U_2(x_2) \sim F[U_1(x_1/M_2)] \]
General rules for applying the Fourier transform in optics

Situation 1: From an object to a plane “really far away”

\[ U_1(x_1) \quad \rightarrow \quad U_2(x_2) \sim F[U_1(x_1/M_1)] \]

Situation 2: From an object to the back focal plane of the microscope objective lens

\[ U_1(x_1) \quad \rightarrow \quad U_2(x_2) \sim F[U_1(x_1/M_2)] \]

Situation 3: From an object to a plane 1 focal length away from a lens (1f-1f system)

\[ U_1(x_1) \quad \rightarrow \quad U_2(x_2) \sim F[U_1(x_1/M_3)] \]
A more exact model: the 4f optical system

\[ E_s(x_s, y_s, 0) \]

\[ E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d) \]

2D Fourier Transform

2D inverse Fourier Transform
A more exact model: the 4f optical system

The Fourier plane provides a measure of the ray angles at the image plane.

2D Fourier Transform

Image plane ray angle $\theta$
A more exact model: the 4f optical system

The Fourier plane provides a measure of the **ray angles at the image plane**

Doesn’t contain info about spatial distribution light

Image plane ray angle $\theta$

2D Fourier Transform

Shift point source

f

f
A more exact model: the 4f optical system

The Fourier plane provides a measure of the **ray angles at the image plane**

Rays are leaving image plane at +20 degrees

Image plane ray angle $\theta$

2D Fourier Transform

Shift point source

20 degrees

$f$

$f$
A more exact model: the 4f optical system

The Fourier plane provides a measure of the ray angles at the image plane.

Rays are coming in at +20 degrees and -15 degrees.

Image plane ray angle $\theta$.

2D Fourier Transform
You typically go between 4 functions to describe one imaging system:

- Coherent point-spread function: $h(x)$
- Incoherent point-spread function: $h_i(x)$
- Coherent transfer function: $H(f_x)$
- Incoherent transfer function: $H_i(f_x)$