

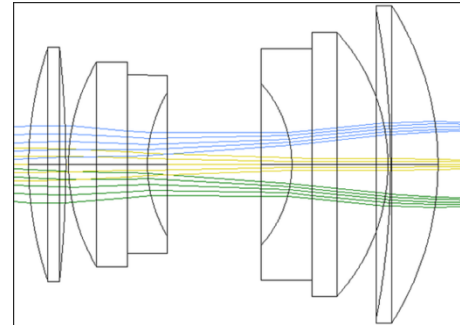
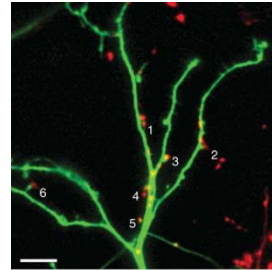
# Lecture 15: Wave optics and Fourier optics

Machine Learning and Imaging

BME 548L  
Roarke Horstmeyer

# What is light and how can we model it?

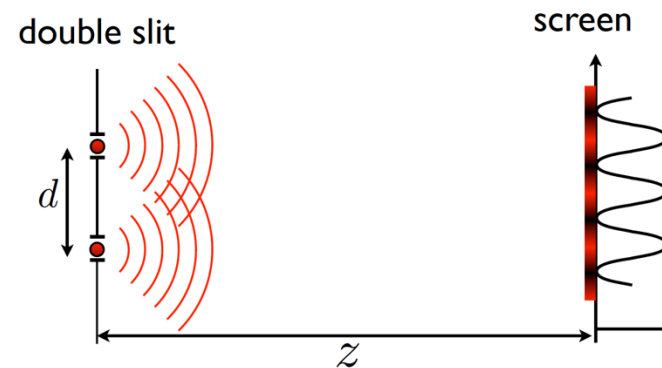
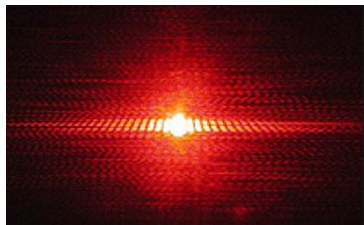
- Interpretation #1: Radiation (*Incoherent*)
- Model: Rays



- Real, non-negative
- Models absorption and brightness

$$I_{\text{tot}} = I_1 + I_2$$

- Interpretation #2: Electromagnetic wave (*Coherent*)
- Model: Waves

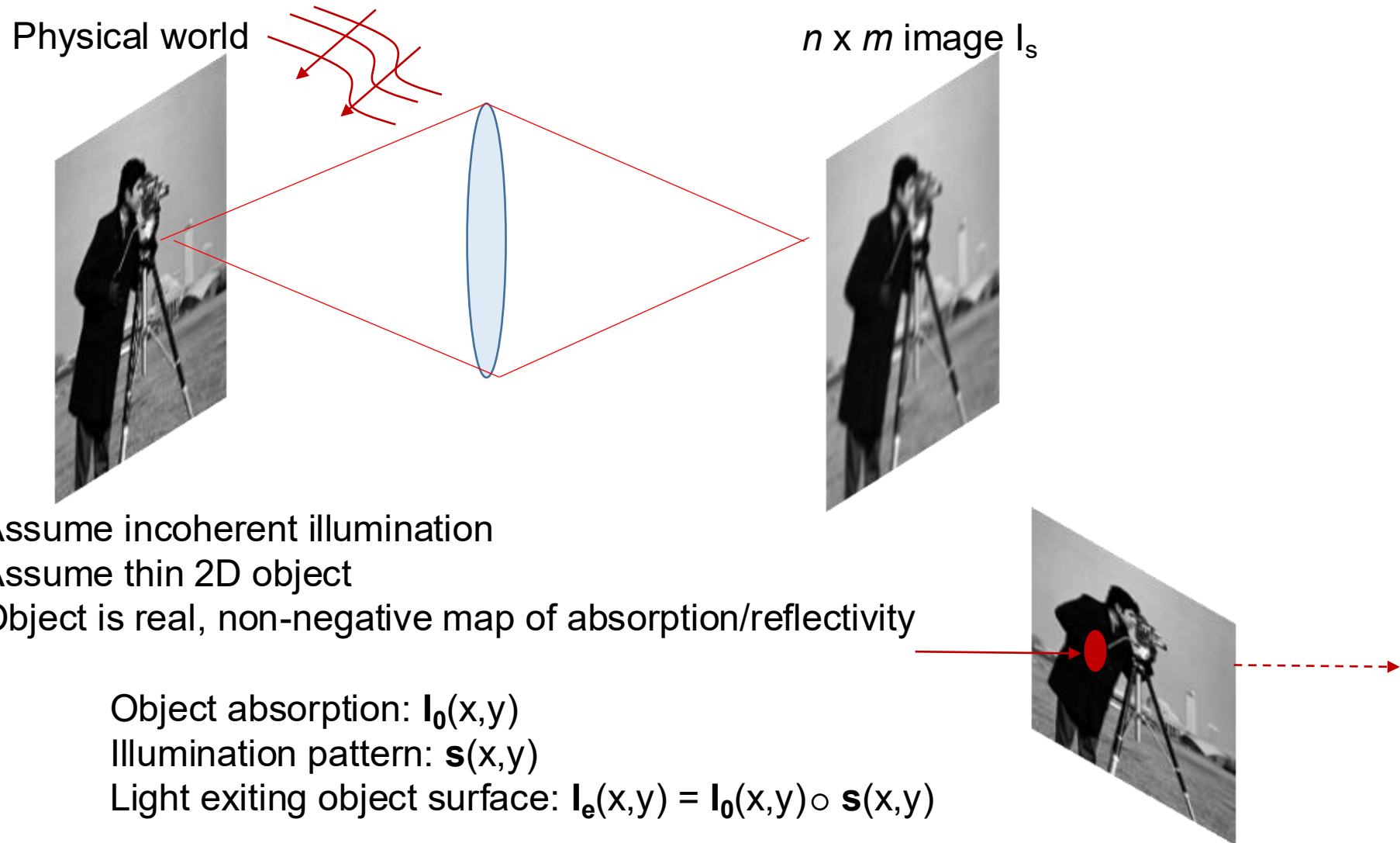


- Complex field
- Models Interference

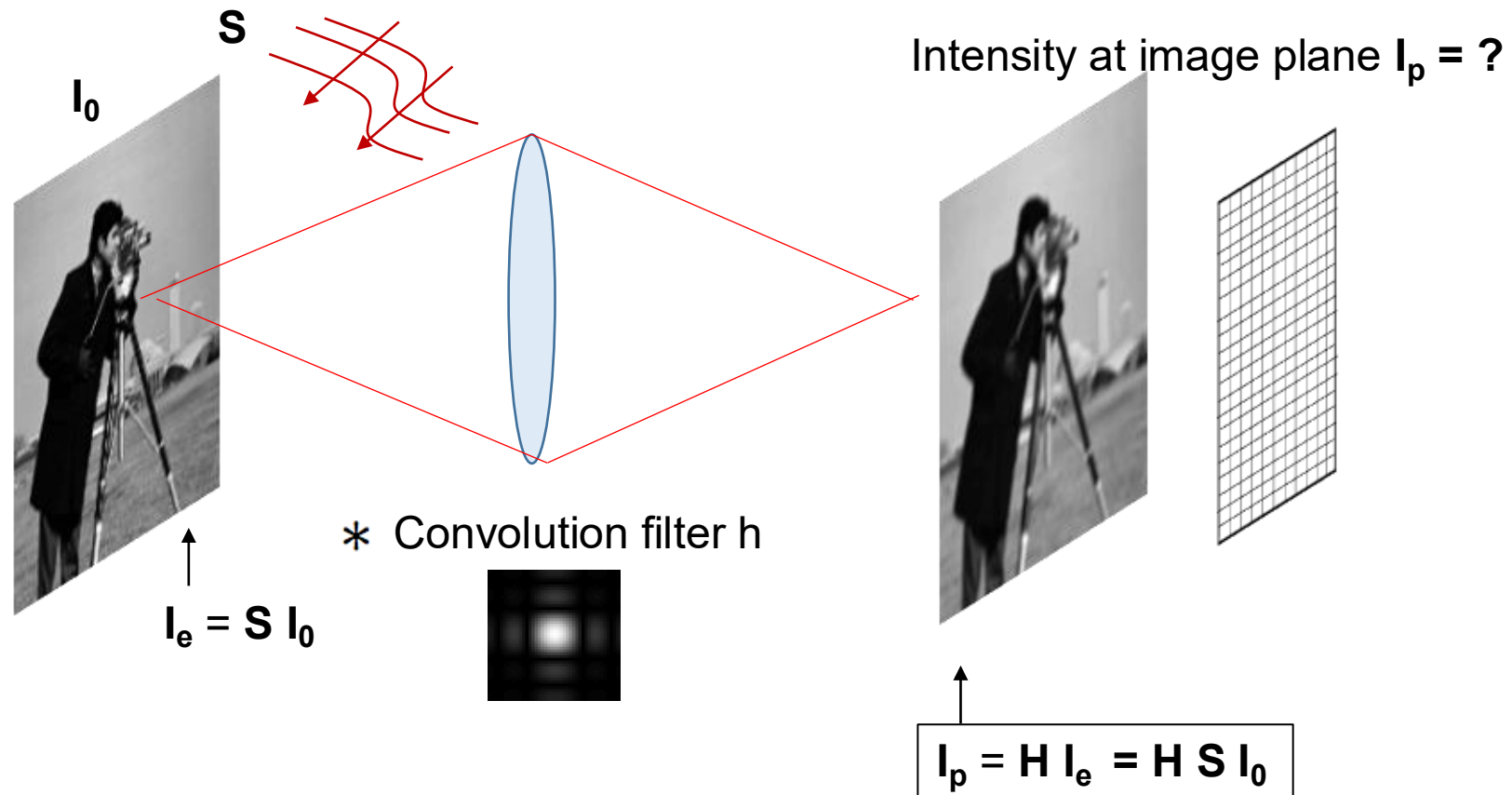
$$E_{\text{tot}} = E_1 + E_2$$

- Interpretation #3: Particle
- Model: Photons

# Simple mathematical model of incoherent image formation



# Simple mathematical model of incoherent image formation



## Summary: simple physical layers for incoherent imaging

- Deal with sample/image intensities  $I$ , real and non-negative

- Effect of illumination is element-wise multiplication  $\lambda$   $I_e(x,y) = \mathbf{S} I_o(x,y)$

- Imaging systems blur the object via point-spread function matrix  $\mathbf{H}$

$$I_b(x,y) = \mathbf{H} I_o(x,y)$$

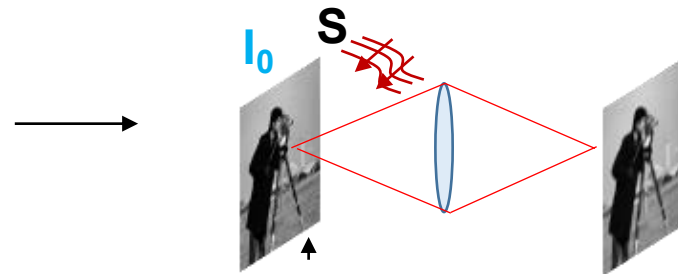
- Discrete pixels down-sample the object via

$$I_d(x,y) = \mathbf{D} I_o(x,y)$$

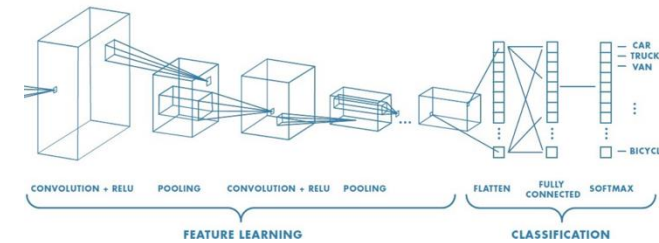
- Add noise into measurement  $I_N(x,y) = \mathbf{D} I_o(x,y) + \mathbf{N}$

- Different colors add linearly  $I_s(x,y) = \sum I_o(x,y,\lambda)$

# Example #1: Optimized illumination pattern (one color)



$[I_e(x, y), y]$

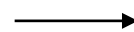


Training data:

$[I_0(x, y), y]$

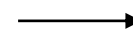
$I_0$ : 100 x 100

Label  $y$ : 1x2



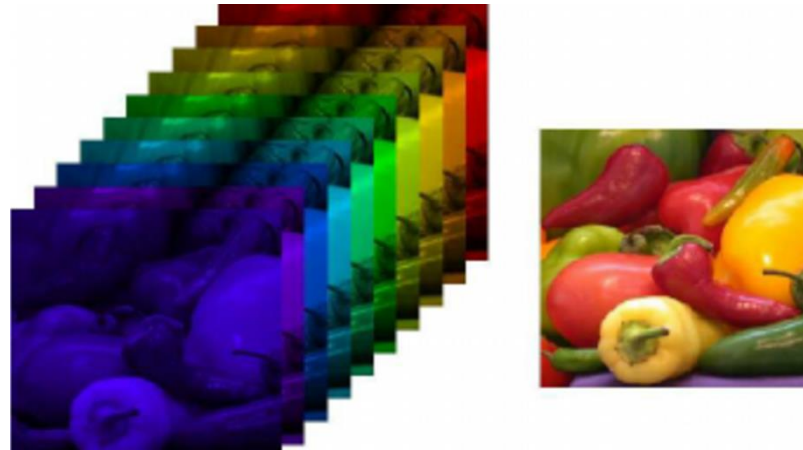
$$I_e(x, y) = S I_0(x, y)$$

Physical Layer

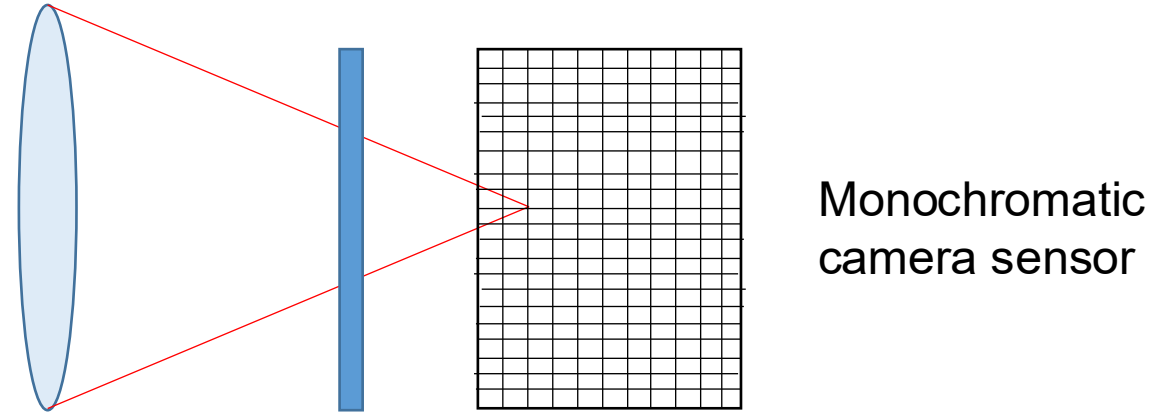
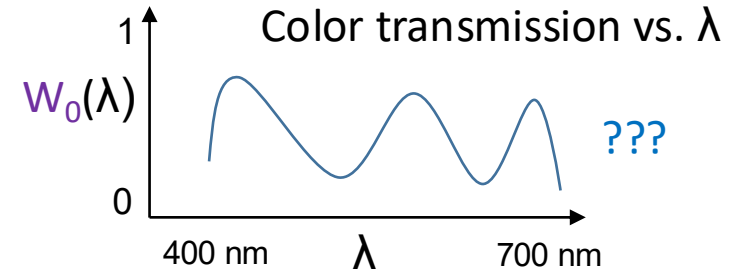


Task =  $W_n \dots \text{ReLU}[W_1 \text{ReLU}[W_0 I_e] \dots]$

# Example #2: Optimized color filter for a grayscale camera



Design optimal color filter for classification:



Training data:

$$[I_0(x, y, \lambda), y]$$

$I_0$ : 100 x 100 pix. x 30

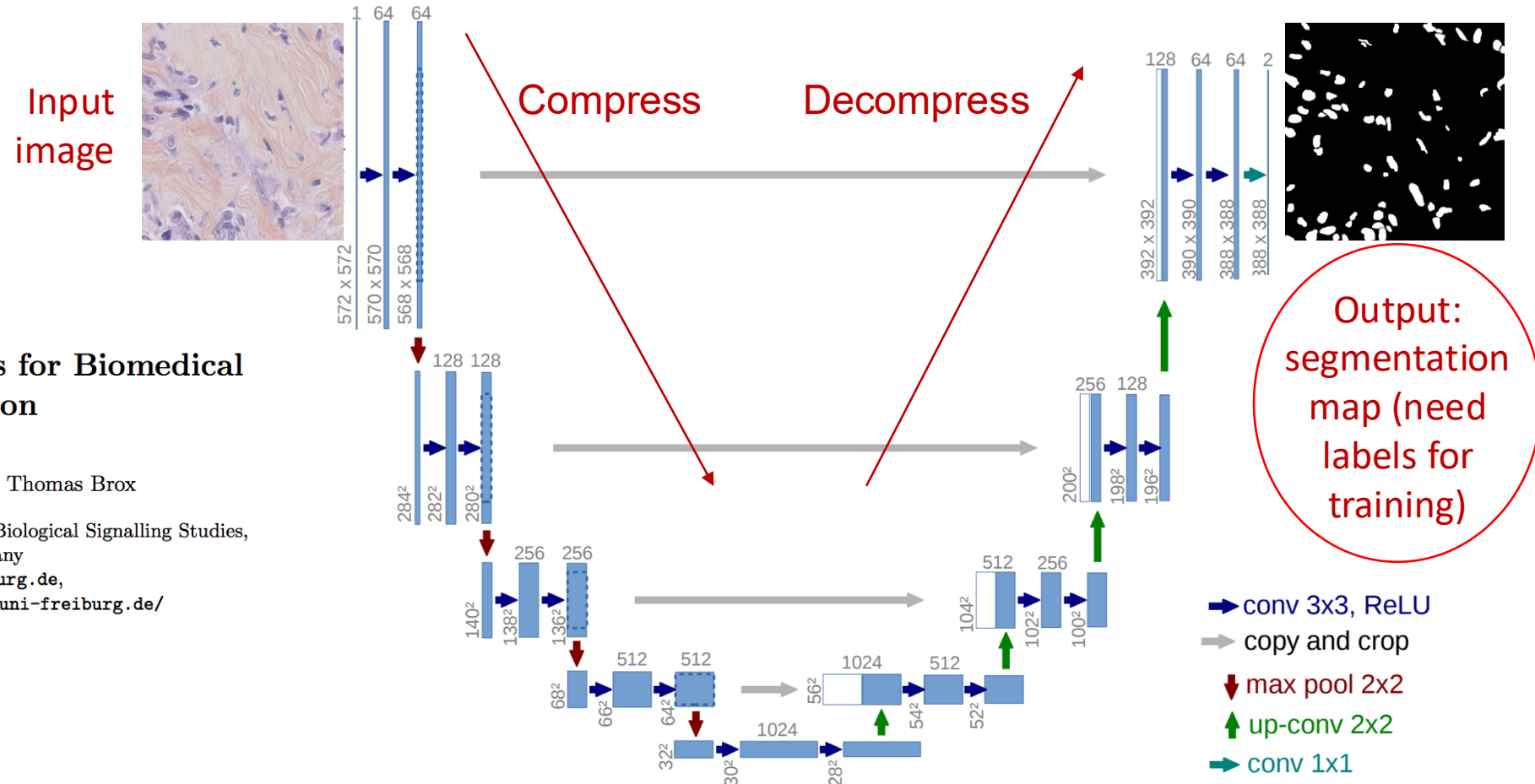
Label  $y$ : 1x3 - pepper, broccoli, green beans

$$I_s(x, y) = \sum_{\lambda} W_0(\lambda) I_0(x, y, \lambda)$$

Physical Layer

# Example 3: learned illumination pattern for improved segmentation

## U-Net Architecture



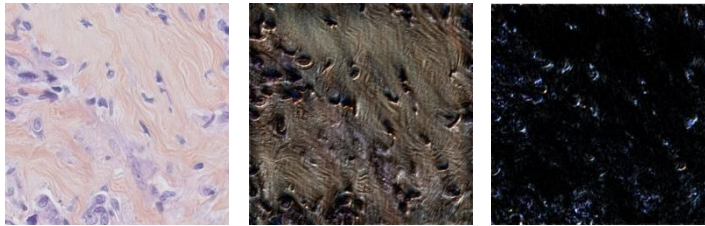
### U-Net: Convolutional Networks for Biomedical Image Segmentation

Olaf Ronneberger, Philipp Fischer, and Thomas Brox

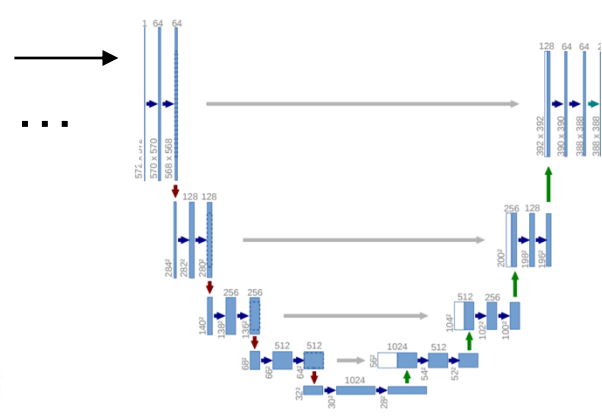
Computer Science Department and BIOS Centre for Biological Signalling Studies,  
 University of Freiburg, Germany  
 ronneber@informatik.uni-freiburg.de,  
 WWW home page: <http://lmb.informatik.uni-freiburg.de/>

# Example 3: learned illumination pattern for improved segmentation

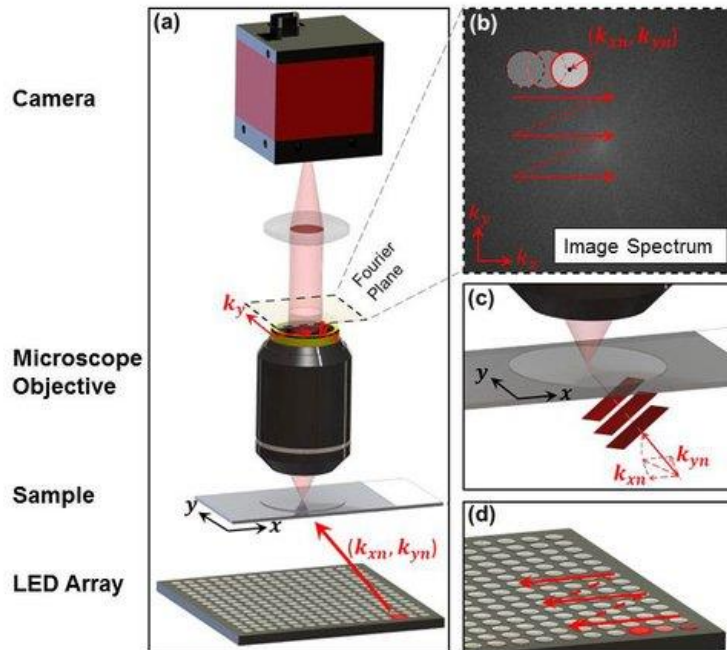
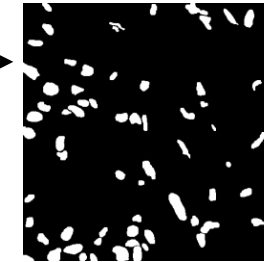
Variably illuminated images from different LEDs



U-Net CNN



Segmentation Map



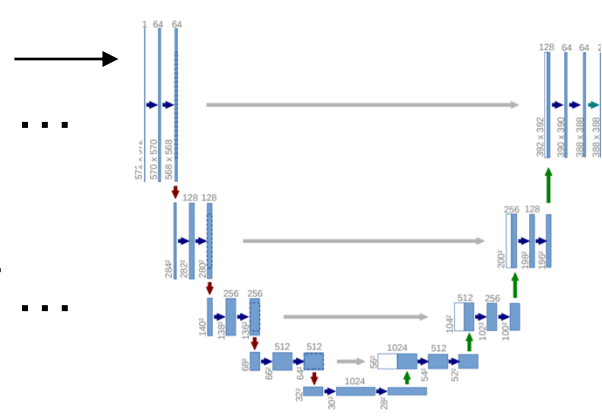
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Variably illuminated images from different LEDs

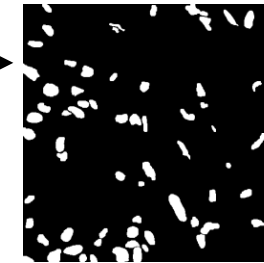


$$w_1 I_1 + w_2 I_2 + w_3 I_3 + \dots$$

U-Net CNN

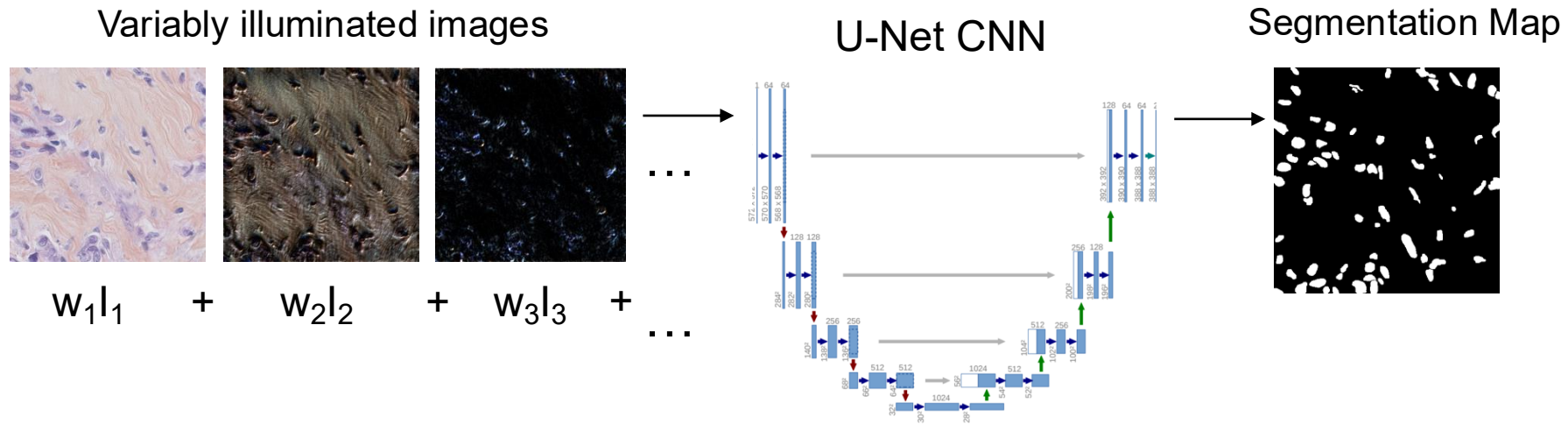


Segmentation Map

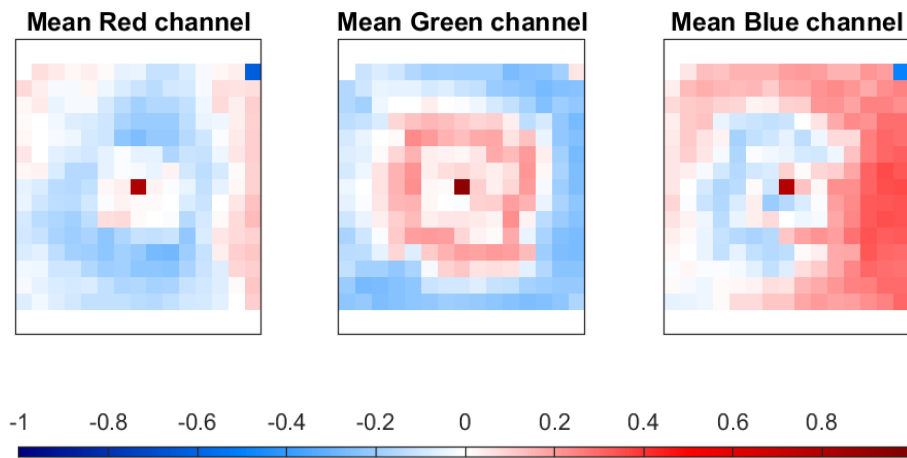


\*If we allow  $w$ 's here to be trainable weights, then we can find ideal brightnesses for different LEDs to illuminate a sample of interest!

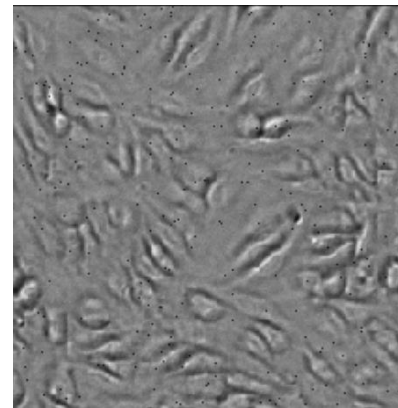
# Example 3: learned illumination pattern for improved segmentation



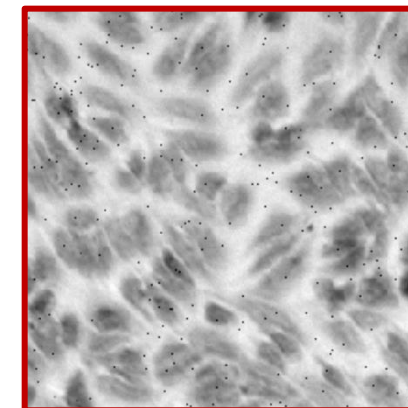
## Optimized illumination for nuclei segmentation



Standard illumination



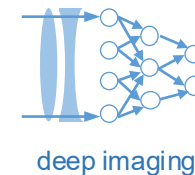
Learned illumination



+5-10% accuracy

See C. Cooke et al., "Physics-enhanced machine learning for virtual fluorescence microscopy," ICCV (2021)

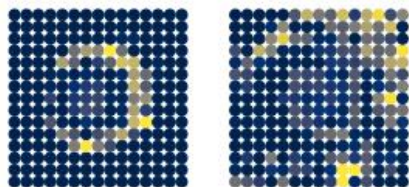
# Multiple Patterns for Fluorescence image inference



1x Pattern Optimization



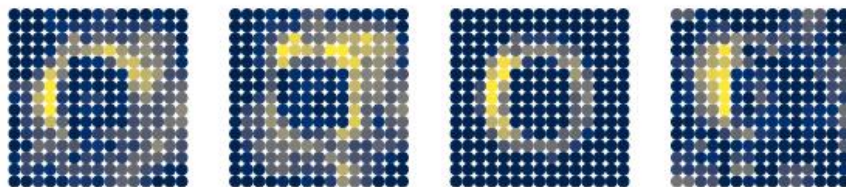
2x Pattern Optimization



4x Pattern Optimization



8x Pattern Optimization

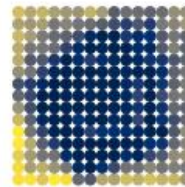
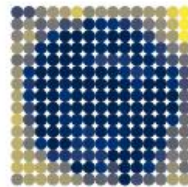
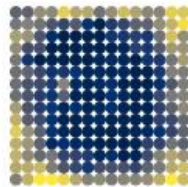
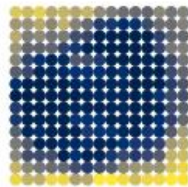


Pattern 1

Pattern 2

Pattern 3

Pattern 4



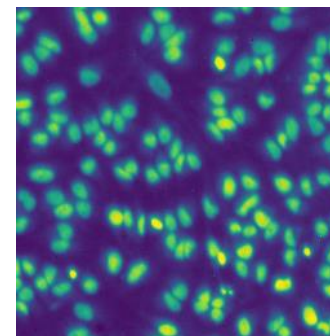
Pattern 5

Pattern 6

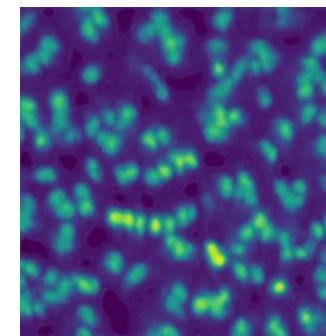
Pattern 7

Pattern 8

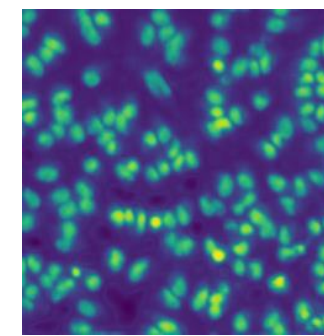
Ground Truth



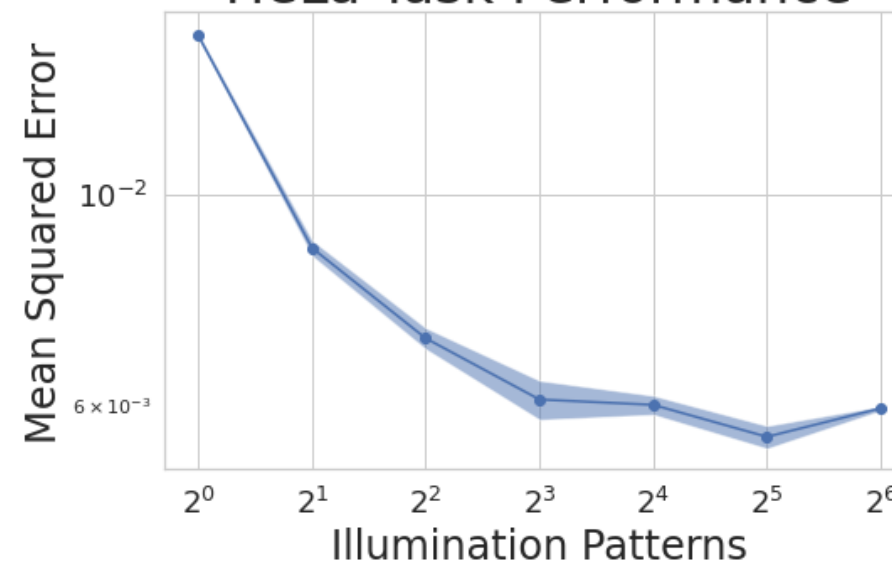
1 Pattern



4 Patterns



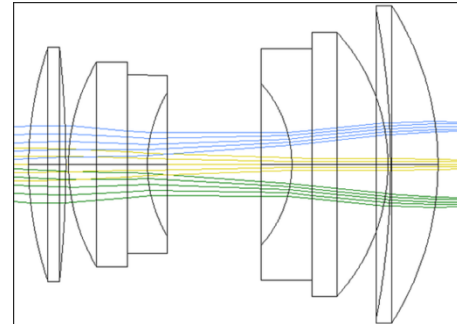
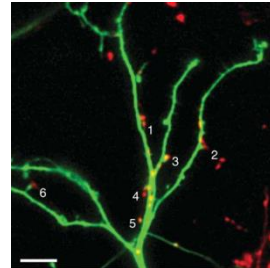
### HeLa Task Performance



See C. Cooke et al., "Physics-enhanced machine learning for virtual fluorescence microscopy," ICCV (2021)

# What is light and how can we model it?

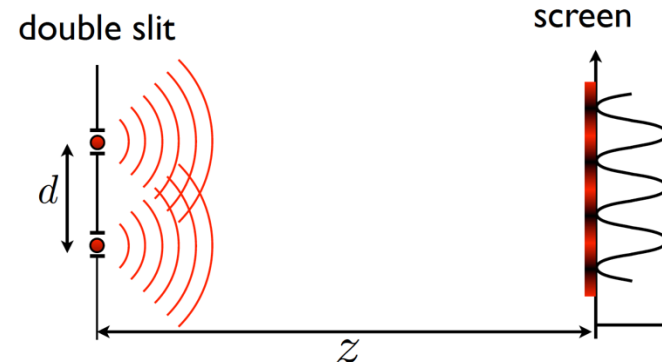
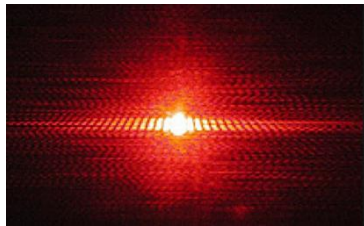
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- Complex field
- Models Interference

$$E_{\text{tot}} = E_1 + E_2$$

**This class: Modeling coherent radiation as a wave**

## Let's take a step back: how does light propagate?

Maxwell's equations  
without any charge

$$\begin{aligned}\nabla \times \vec{\mathcal{E}} &= -\mu \frac{\partial \vec{\mathcal{H}}}{\partial t} \\ \nabla \times \vec{\mathcal{H}} &= \epsilon \frac{\partial \vec{\mathcal{E}}}{\partial t} \\ \nabla \cdot \epsilon \vec{\mathcal{E}} &= 0 \\ \nabla \cdot \mu \vec{\mathcal{H}} &= 0.\end{aligned}$$

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1. Take the curl of both sides of first equation
2. Substitute 2<sup>nd</sup> and 3<sup>rd</sup> equation
3. Arrive at the wave equation:

$$\nabla^2 \vec{\mathcal{E}} - \frac{n^2}{c^2} \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = 0 \quad n = \left( \frac{\epsilon}{\epsilon_0} \right)^{1/2} \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

## Let's take a step back: how does light propagate?

Considering light that isn't pulsed over time, we can use the following solution:

$$u(P, t) = A(P) \cos[2\pi\nu t + \phi(P)]$$

$$u(P, t) = \text{Re}\{U(P) \exp(-j2\pi\nu t)\},$$

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With this particular solution, we get the following important time-independent equation:

Helmholtz  
Equation

$$(\nabla^2 + k^2)U = 0.$$

$$k = 2\pi n \frac{\nu}{c} = \frac{2\pi}{\lambda},$$

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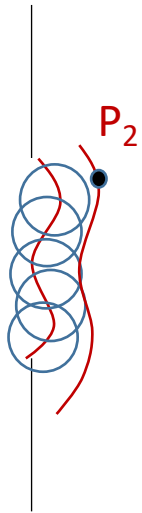
This is an important equation in physics. We won't go into the details, but it leads to the Huygen-Fresnel principle:

$$U(P_2) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{21})}{r_{21}} \cos\theta ds$$

## Plane-to-plane light propagation via the "paraxial approximation"

The Huygens-Fresnel Equation

$$U(P_2) = \frac{1}{j\lambda} \iint_{\Sigma} U(P_1) \frac{\exp(jkr_{21})}{r_{21}} \cos \theta \, ds$$

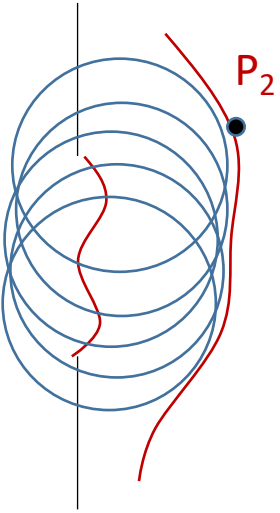


Aperture

## Plane-to-plane light propagation via the "paraxial approximation"

The Huygens-Fresnel Equation

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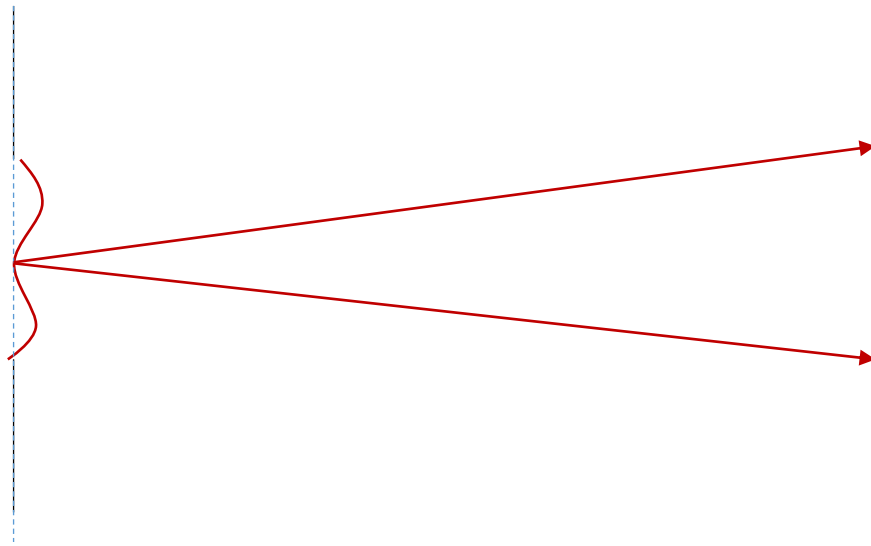
Generally connects two points in 3D:

$$U(P_1) = U(x_1, y_1, z_1)$$

$$U(P_2) = U(x_2, y_2, z_2)$$

## Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):



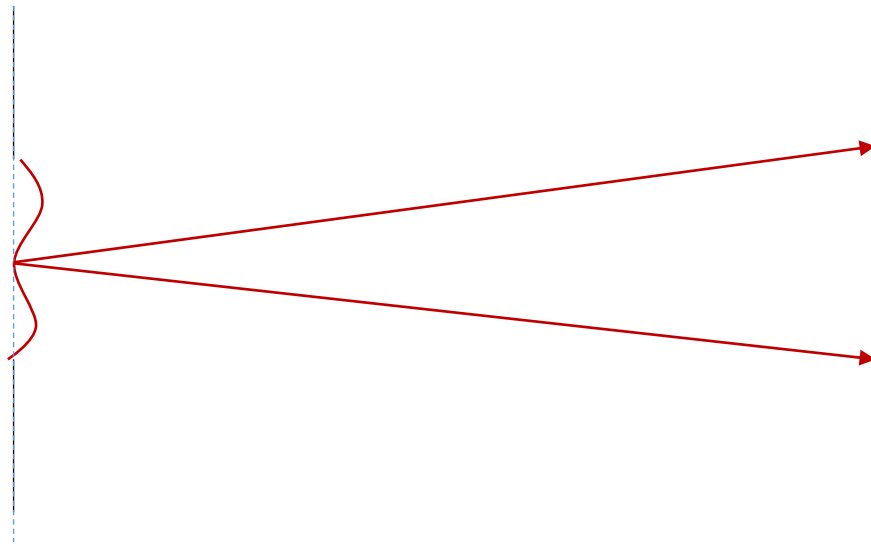
$$U(P_1) = U(x_1, y_1, z_1 = z_{p1})$$

$$U(P_2) = U(x_2, y_2, z_2 = z_{p2})$$

$$U(P) = E(x, y, z)e^{ikz}$$

## Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):



**Paraxial approximation:**

$$\nabla_{\perp}^2 U + 2ik \frac{dU}{dz} = 0$$

$$\nabla_{\perp}^2 \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$U(P_1) = U(x_1, y_1, z_1 = z_{p1})$$

$$U(P_2) = U(x_2, y_2, z_2 = z_{p2})$$

$$U(P) = E(x, y, z)e^{ikz}$$

## Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):

**Paraxial approximation:**

$$\nabla_{\perp}^2 U + 2ik \frac{dU}{dz} = 0 \quad \text{Substitute in } U(P) = E(x, y, z)e^{ikz} \text{ and crank the wheel,}$$

$$\nabla_{\perp}^2 E + 2ik \frac{dE}{dz} + 2k^2 E = 0 \quad \text{Paraxial Helmholtz Equation. This has an exact integral solution:}$$

## Plane-to-plane light propagation via the "paraxial approximation"

We are usually concerned about propagation between two planes (almost always in an optical system):

**Paraxial approximation:**

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$$\nabla_{\perp}^2 E + 2ik \frac{dE}{dz} + 2k^2 E = 0 \quad \text{Paraxial Helmholtz Equation. This has an exact integral solution:}$$

$$E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z} [(x-x')^2 + (y-y')^2]} dx' dy'$$

Fresnel diffraction  
integral

**This is how light propagates from one plane to the next. It's a convolution!**

## Fresnel light propagation as a convolution

$$E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z} [(x-x')^2 + (y-y')^2]} dx' dy'$$

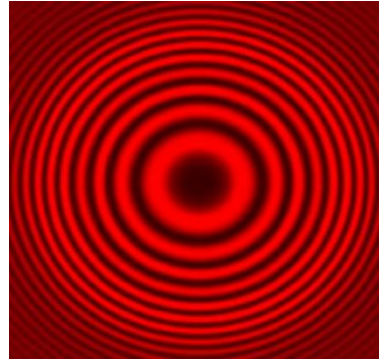
$$h(x, y, z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z} (x^2 + y^2)}$$

$$E(x, y, z) = E(x, y, 0) * h(x, y, z)$$

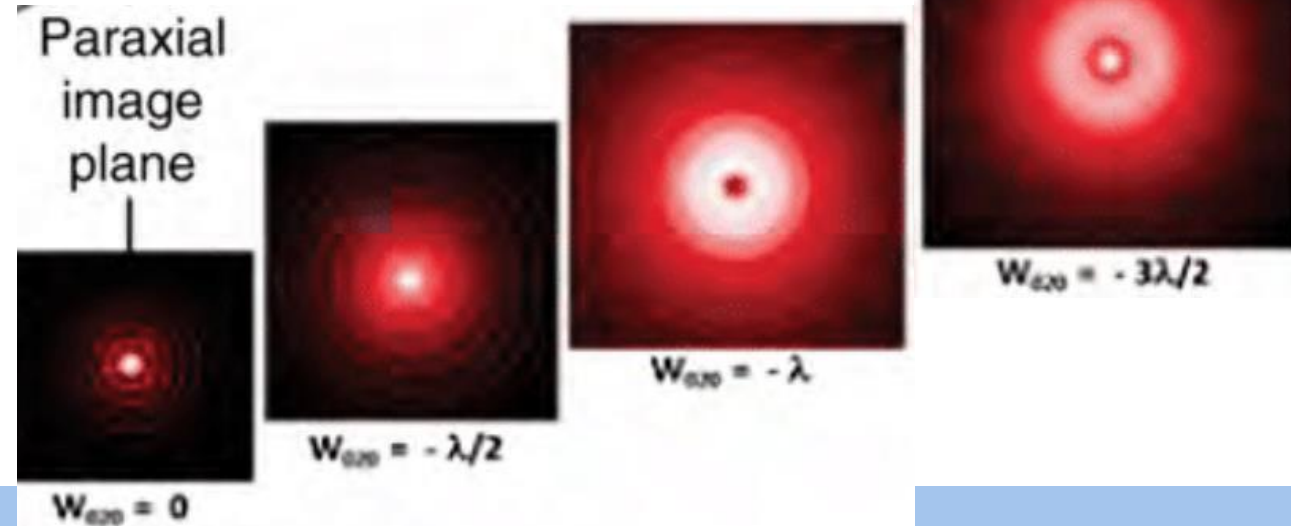
## Fresnel light propagation as a convolution

$$E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z} [(x-x')^2 + (y-y')^2]} dx' dy'$$

$$h(x, y, z) = \frac{e^{ikz}}{i\lambda z} e^{i\frac{k}{2z} (x^2 + y^2)}$$



$$E(x, y, z) = E(x, y, 0) * h(x, y, z)$$



## From the Fresnel approximation to the Fraunhofer approximation

**Fresnel Approximation:**

$$E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z} [(x-x')^2 + (y-y')^2]} dx' dy'$$

Lets assume that the second plane is “pretty far away” from the first plane. Then,

$$z > \frac{2D^2}{\lambda}$$

## From the Fresnel approximation to the Fraunhofer approximation

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1. Expand the squaring

$$E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint E(x', y', 0) e^{\frac{ik}{2z} (x^2 + y^2)} e^{\frac{ik}{2z} (x'^2 + y'^2)} e^{\frac{ik}{2z} (xx' + yy')} dx' dy'$$

## From the Fresnel approximation to the Fraunhofer approximation

### Fresnel Approximation:

$$E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint_{-\infty}^{+\infty} E(x', y', 0) e^{\frac{ik}{2z} [(x-x')^2 + (y-y')^2]} dx' dy'$$

Lets assume that the second plane is “pretty far away” from the first plane. Then,

$$z > \frac{2D^2}{\lambda}$$

1. Expand the squaring

$$E(x, y, z) = \frac{e^{ikz}}{i\lambda z} \iint E(x', y', 0) e^{\frac{ik}{2z} (x^2 + y^2)} e^{\frac{ik}{2z} (x'^2 + y'^2)} e^{\frac{ik}{2z} (xx' + yy')} dx' dy'$$

2. Front term comes out, assume second term goes away, then,

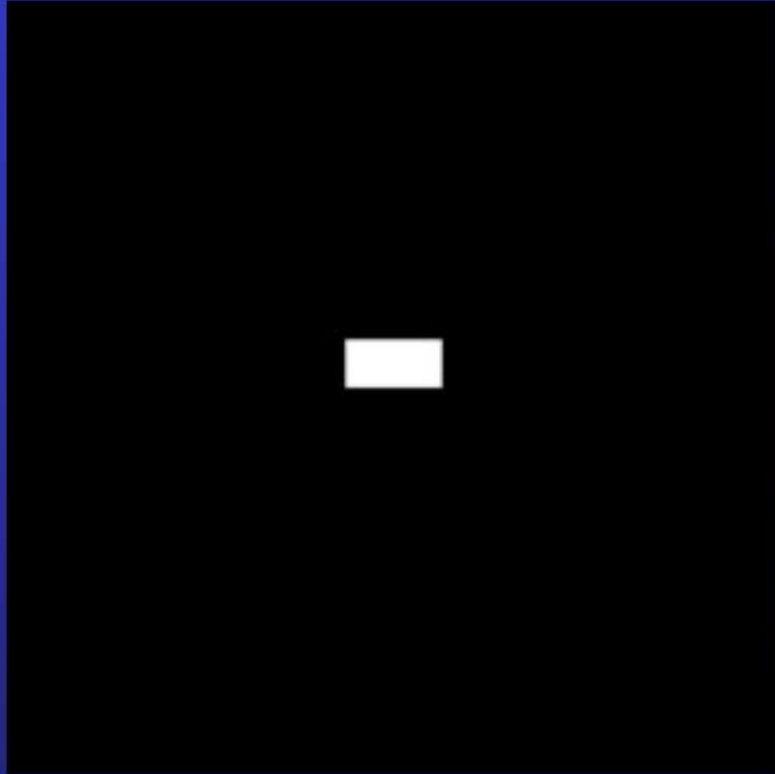
$$E(x, y, z) = C \iint E(x', y', 0) e^{\frac{ik}{2z} (xx' + yy')} dx' dy'$$

$$C = \frac{e^{ikz}}{i\lambda z} e^{\frac{ik}{2z} (x^2 + y^2)}$$

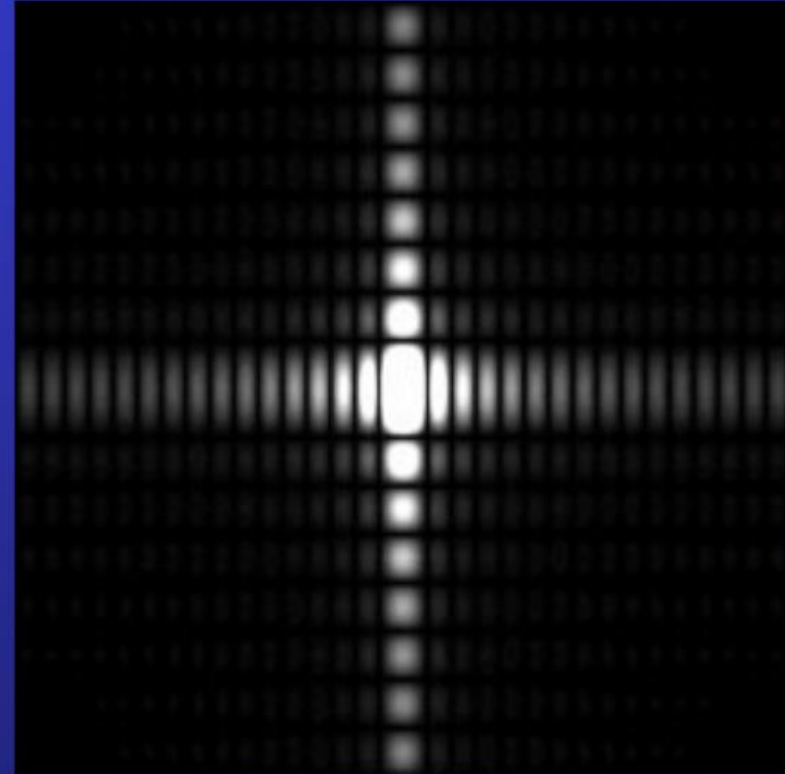
**Fraunhofer diffraction is a Fourier transform!!!!!!!**

This is the aperture

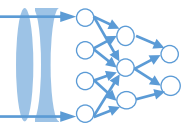
This is what you see far away



Two-dimensional rectangle function as an image



d) Magnitude of Fourier spectrum of the 2-D rectangle



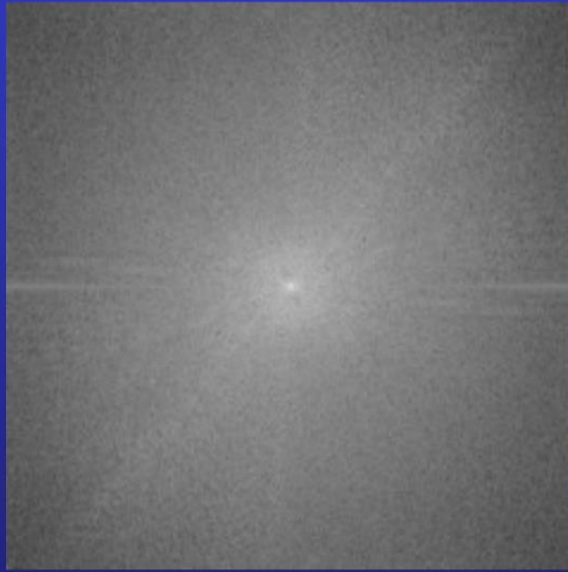
deep imaging



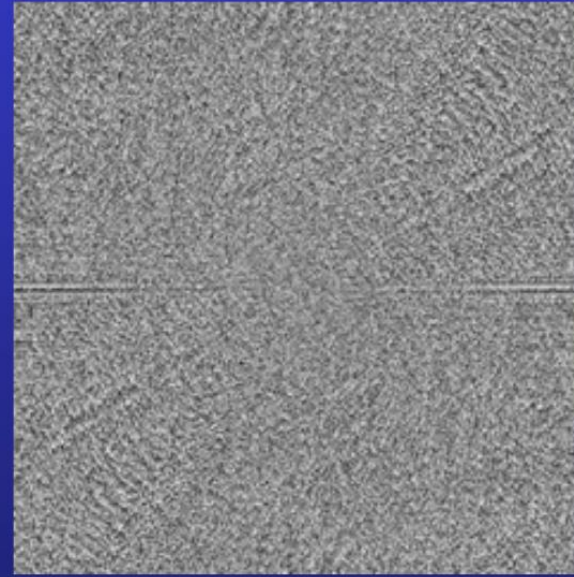
**Cheetah**



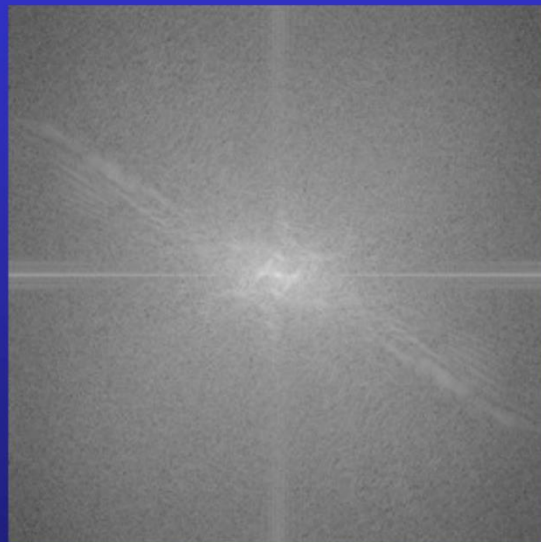
**Zebra**



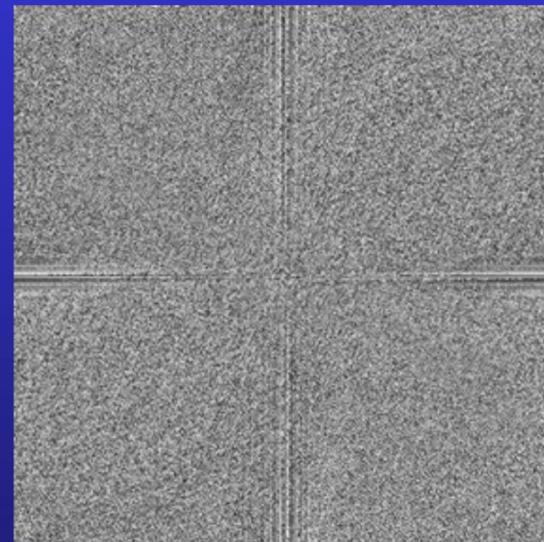
**magnitude of cheetah**



**phase of cheetah**

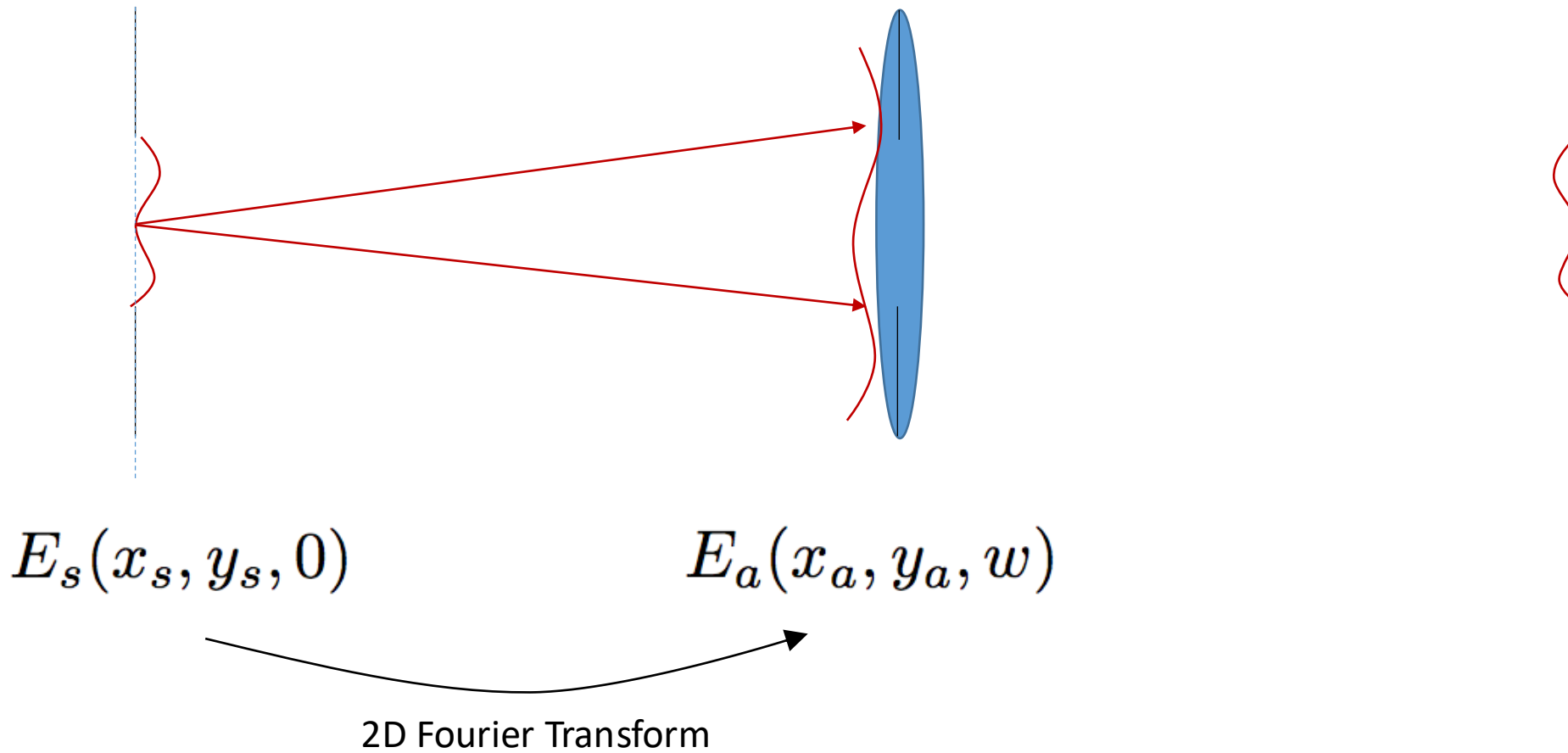


**magnitude of zebra**

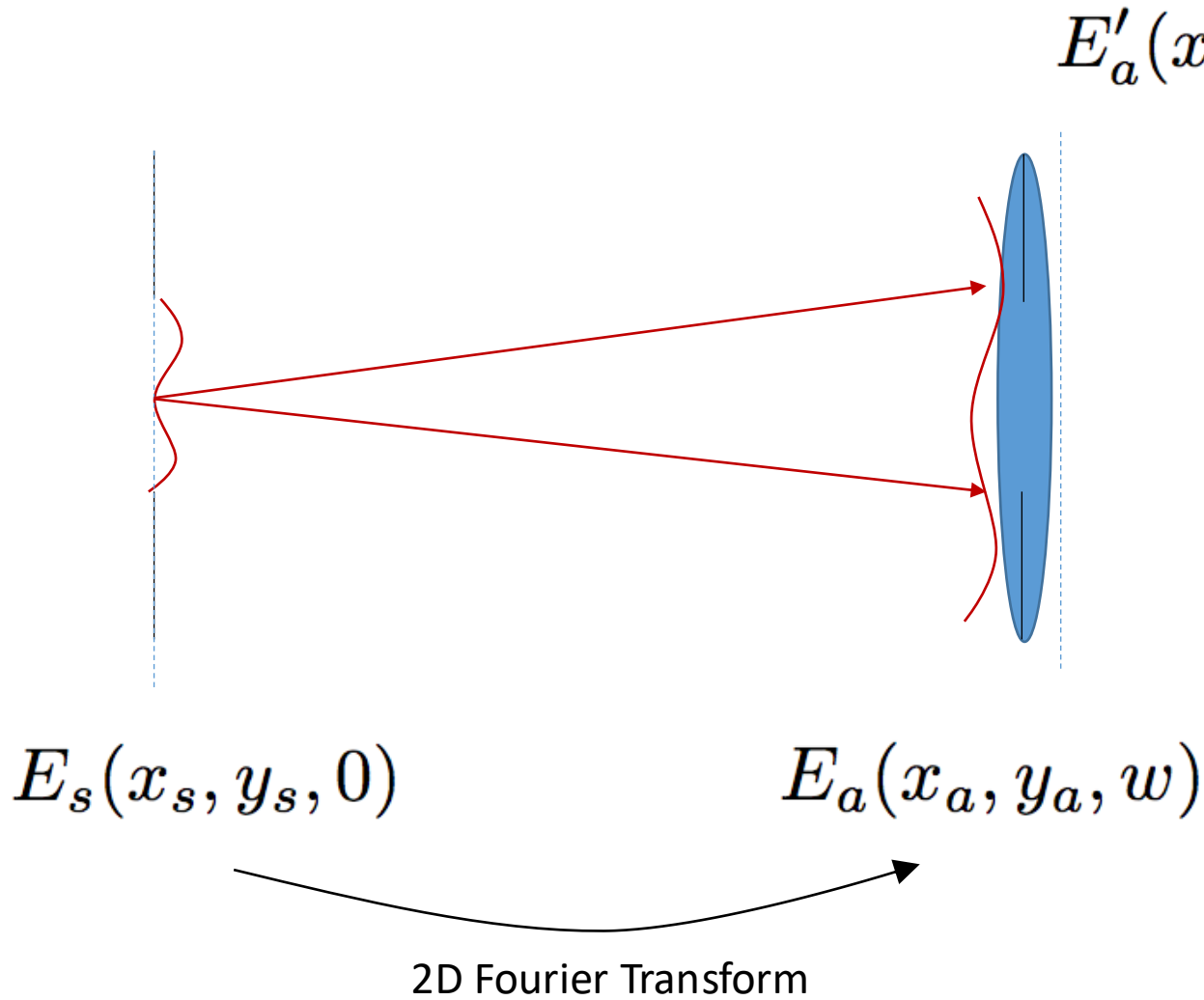


**phase of zebra**

# Model of a microscope (or camera) using Fourier transforms:



# Model of a microscope (or camera) using Fourier transforms:



Effect of the lens is to block light.

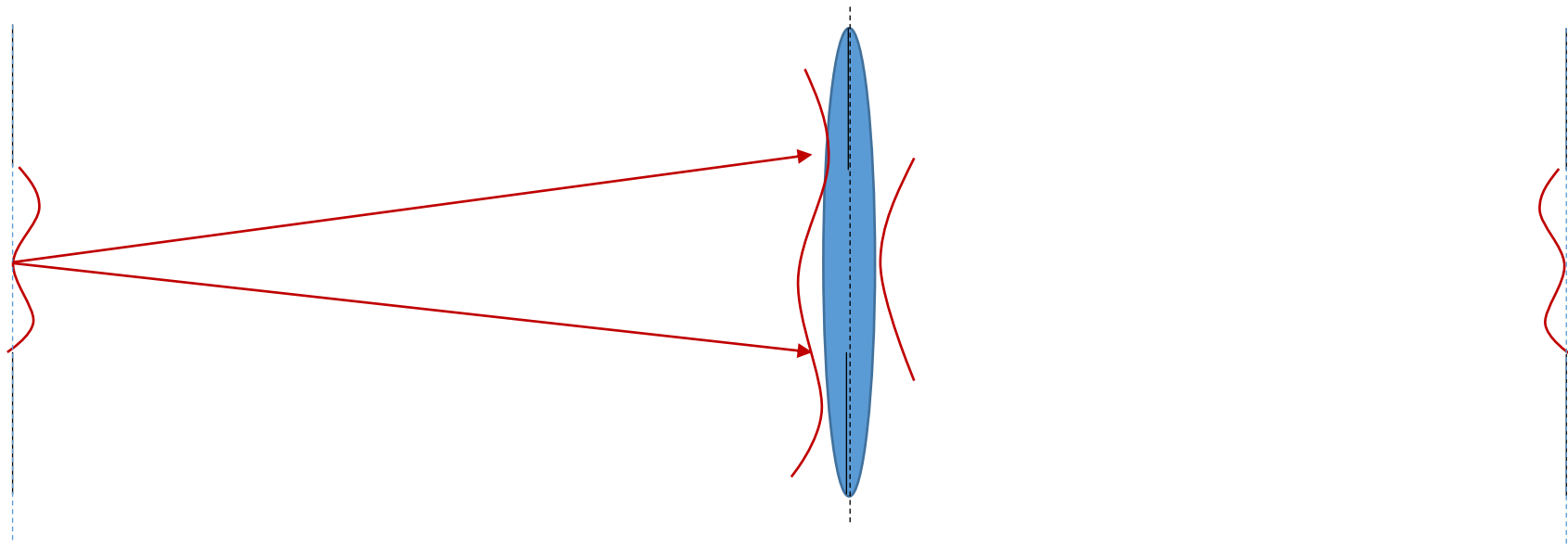
Use *thin object approximation* to determine distribution of light on the immediate other side of the lens stop:

$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$



# Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?



$$E_s(x_s, y_s, 0)$$

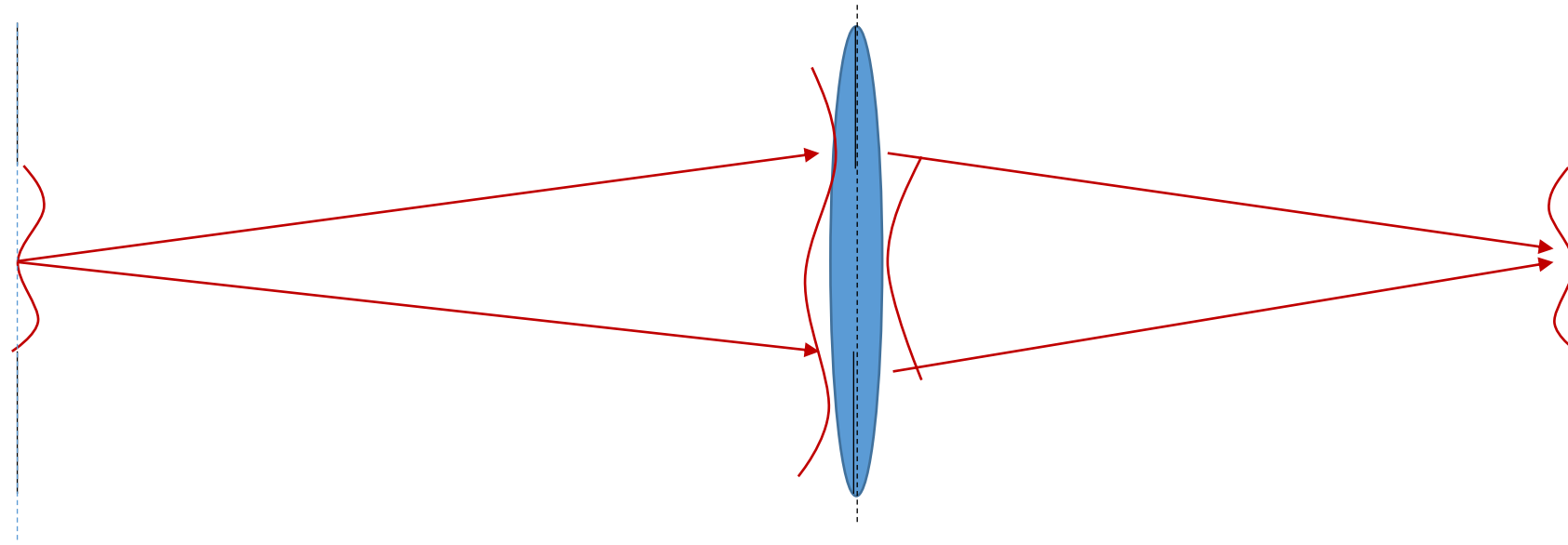
$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

2D Fourier Transform  $\rightarrow$

# Model of a microscope (or camera) using Fourier transforms:

Last piece of the puzzle: what happens from lens to sensor?

***inverse Fourier transform!***



$$E_s(x_s, y_s, 0)$$

$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

2D Fourier Transform

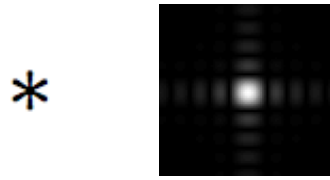
2D inverse Fourier Transform

# This process should sound familiar....

Input image  
 $U_1(x,y)$



Convolution filter  $h$



=

Output image  
 $U_2(x,y)$

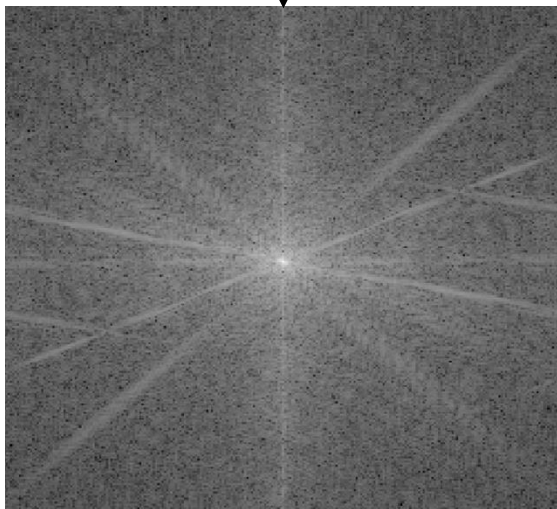


$F[U_1]$

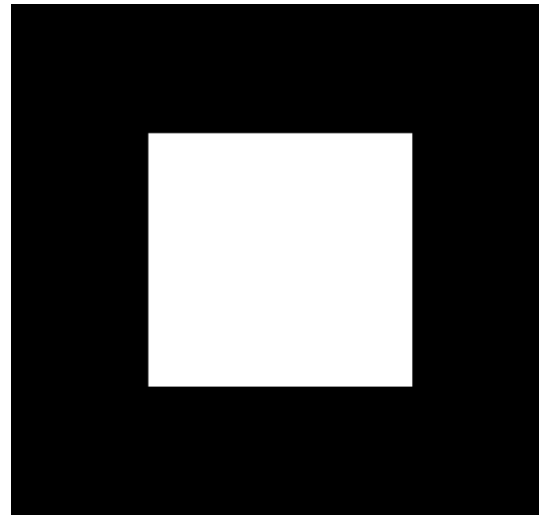
$F[h]$

$F^{-1}[H\hat{U}_1]$

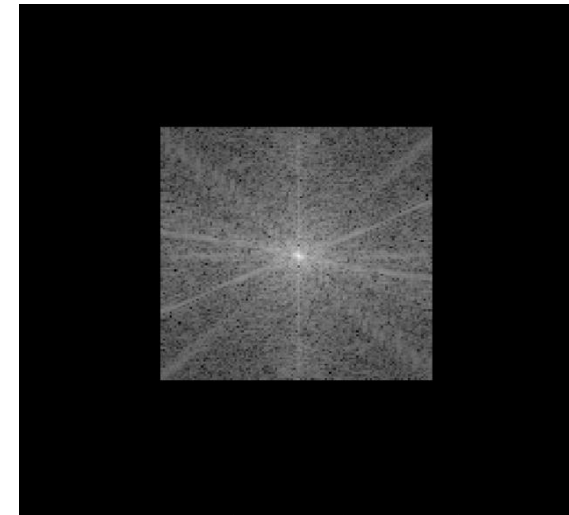
Input spectrum  
 $\hat{U}_1(f_x, f_y)$



$\cdot$

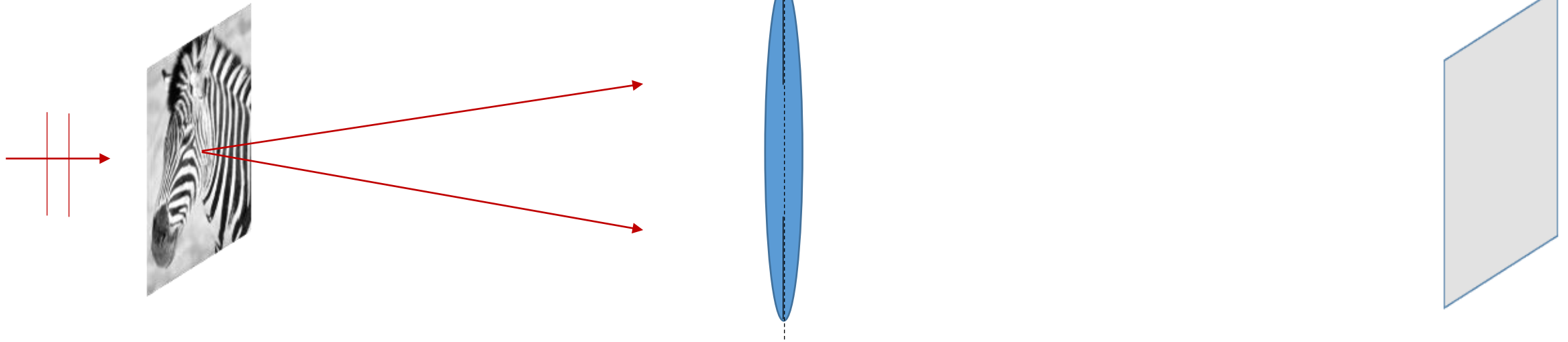


=



# Model of image formation for wave optics (coherent light):

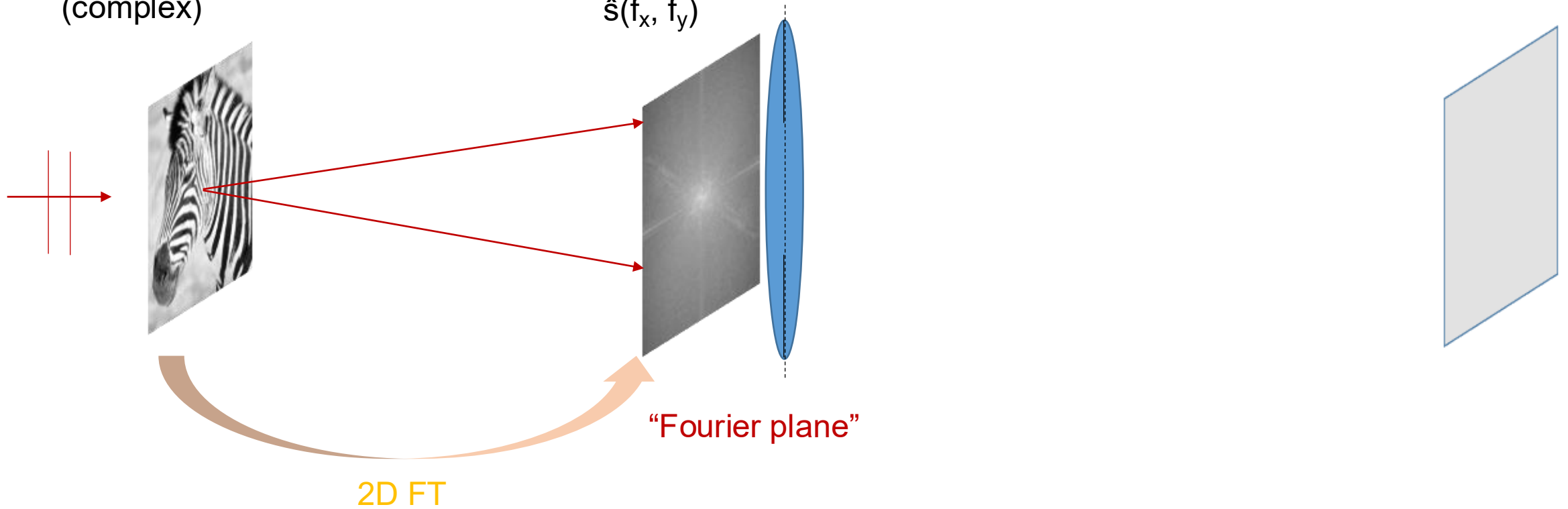
1. Discrete sample function  $s(x,y)$  (complex)



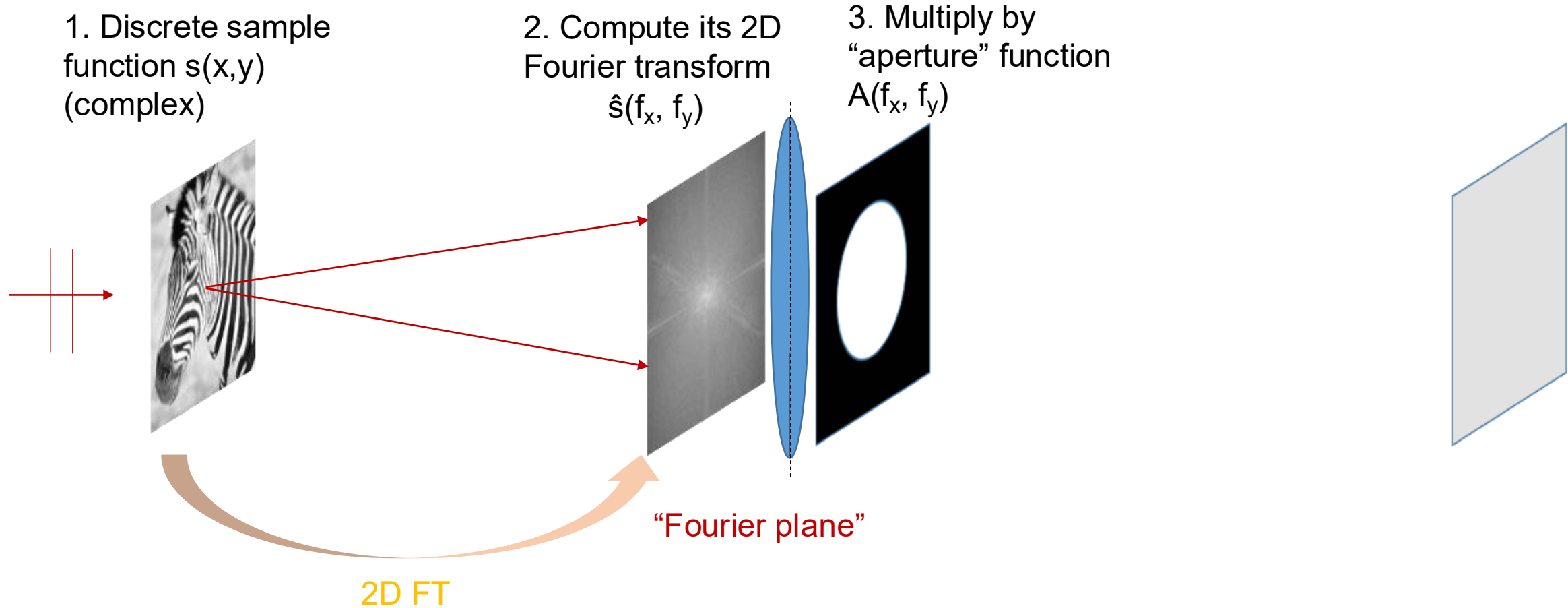
# Model of image formation for wave optics (coherent light):

1. Discrete sample function  $s(x,y)$  (complex)

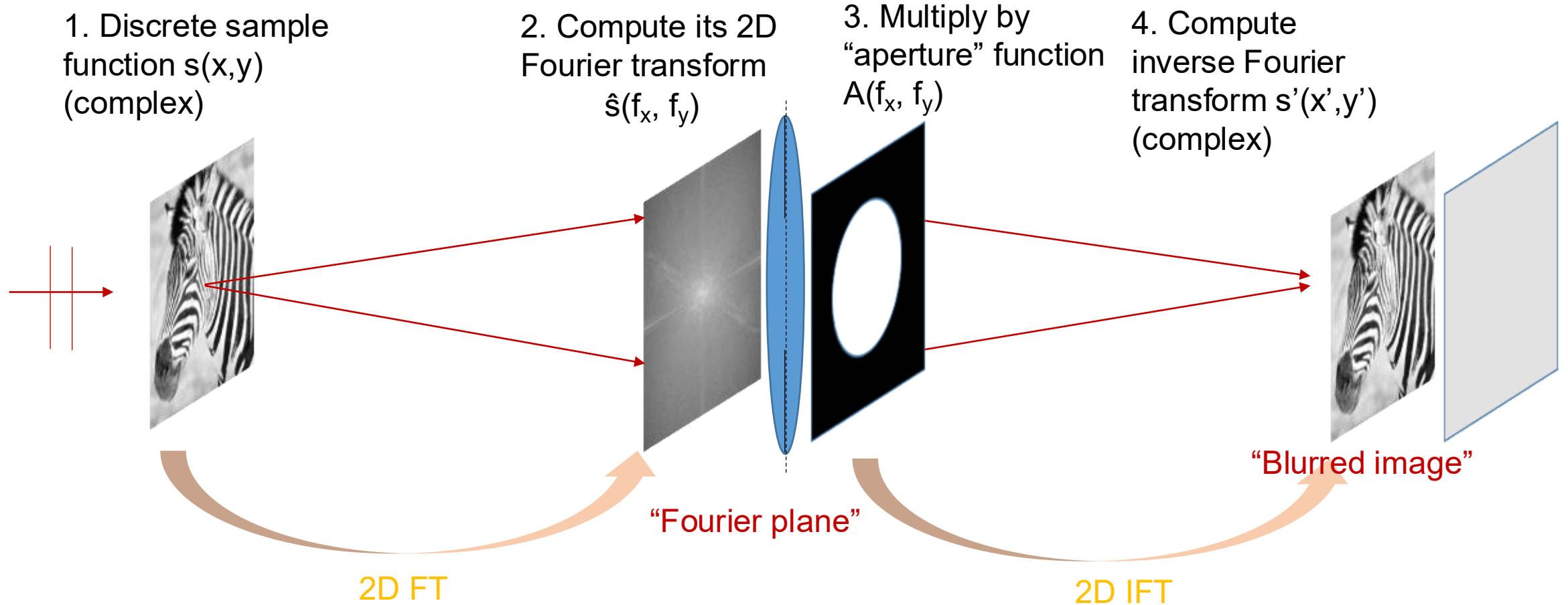
2. Compute its 2D Fourier transform  $\hat{s}(f_x, f_y)$



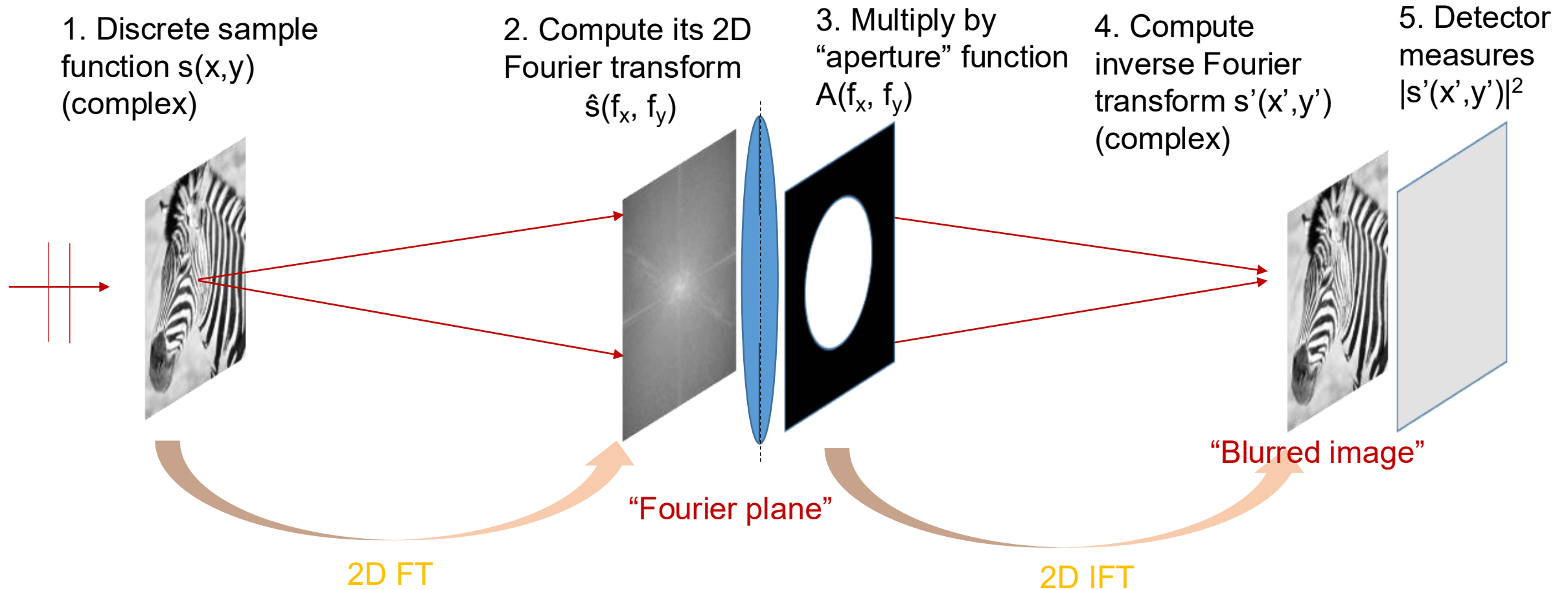
# Model of image formation for wave optics (coherent light):



# Model of image formation for wave optics (coherent light):



# Model of image formation for wave optics (coherent light):

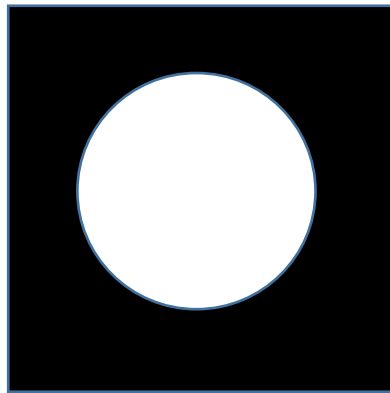


# Can also model this using the Convolution Theorem

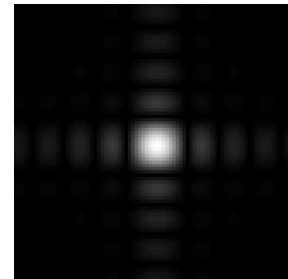
Aperture function (lens shape)

Camera blur function (IFT of lens shape)

$A(f_x, f_y)$



$h(x, y)$

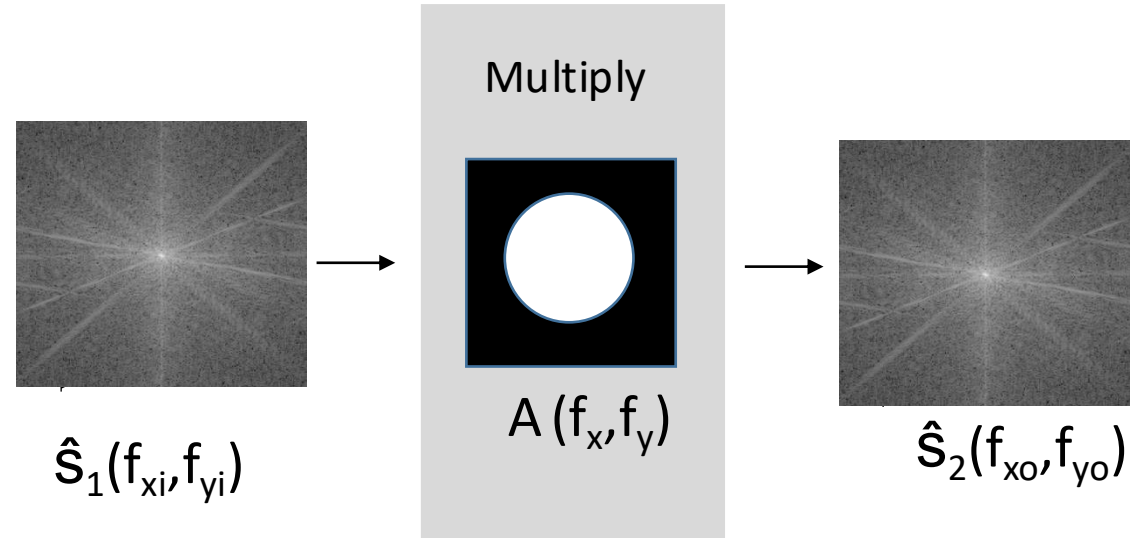


2D IFT

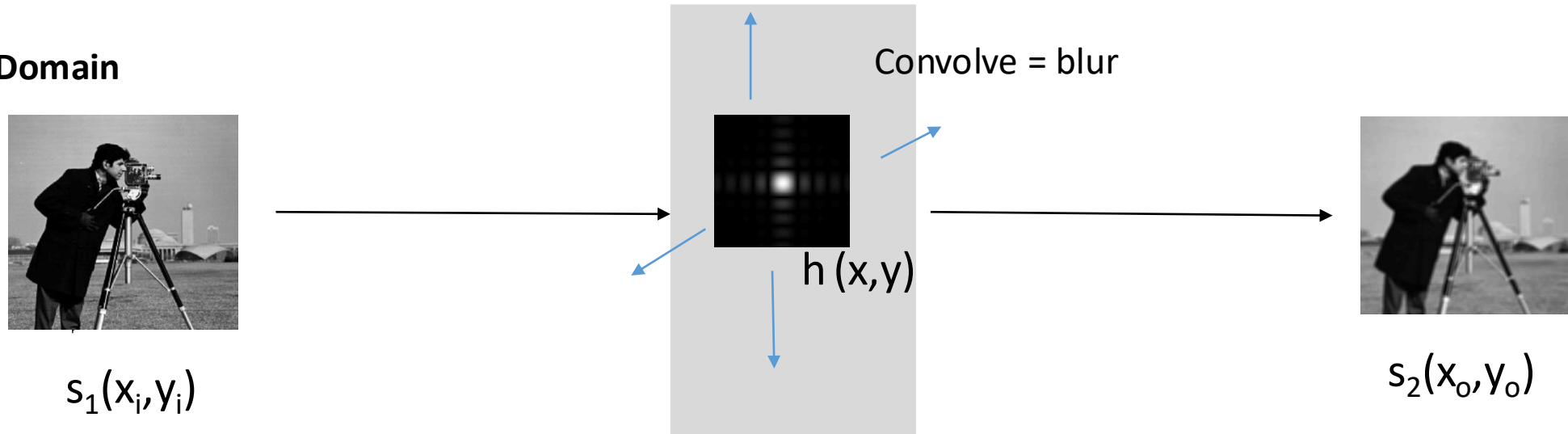


# Two modeling choices for the camera:

## Spatial Frequency Domain



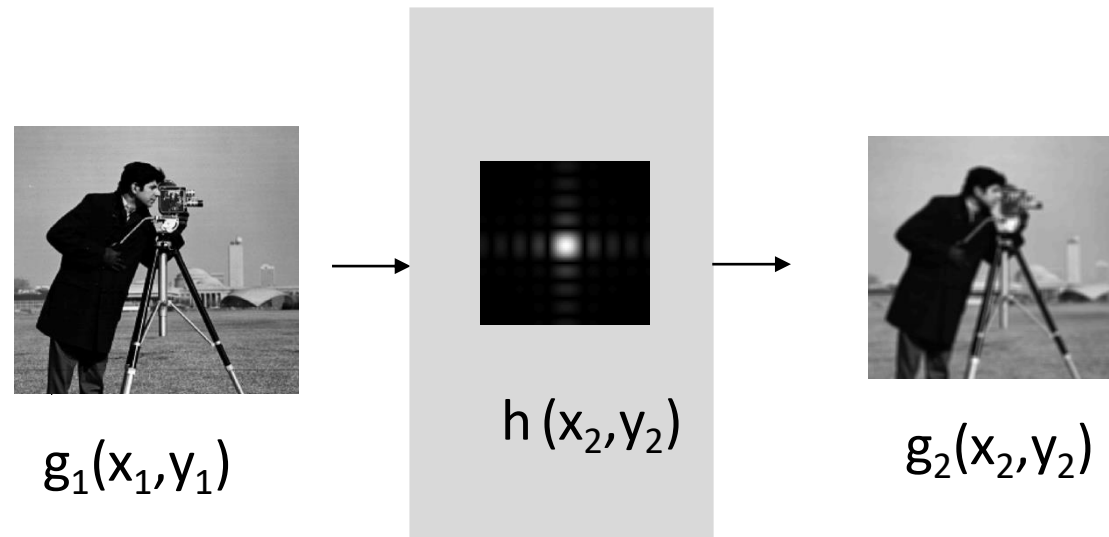
## Spatial Domain



## Linear systems and the black box

### The optical black box system and the point-spread function:

Light  $g_1(x_i, y_i)$  entering “black box” optical system modified by system point-spread function

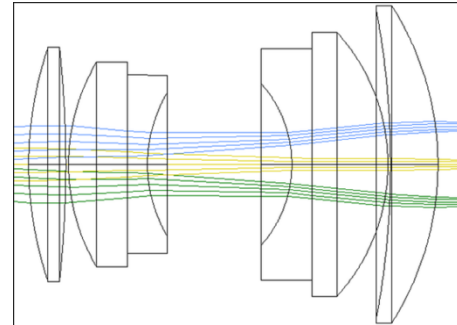
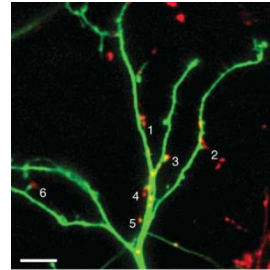


$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(x_1, y_1) h(x_2 - x_1, y_2 - y_1) dx_1 dy_1$$

**Assume shift invariance:  
This is the system point-spread function**

# Summary of two models for image formation

- Interpretation #1: Radiation (*Incoherent*)
- Model: Rays



- Real, non-negative
- Models absorption and brightness

$$I_{\text{tot}} = I_1 + I_2$$

# Mathematical model of for incoherent image formation

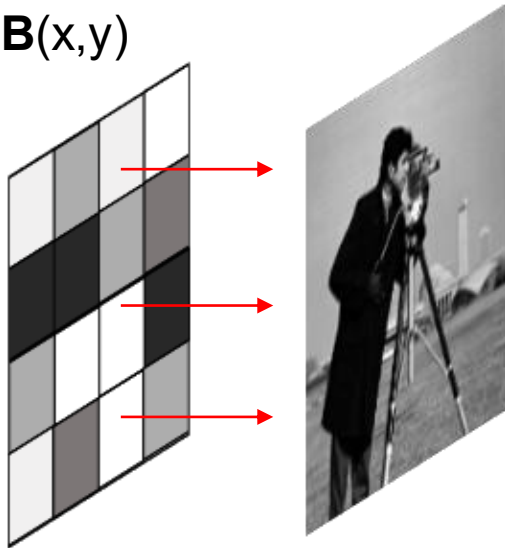
- All quantities are real, and non-negative

Object absorption:

$$S_0(x,y)$$

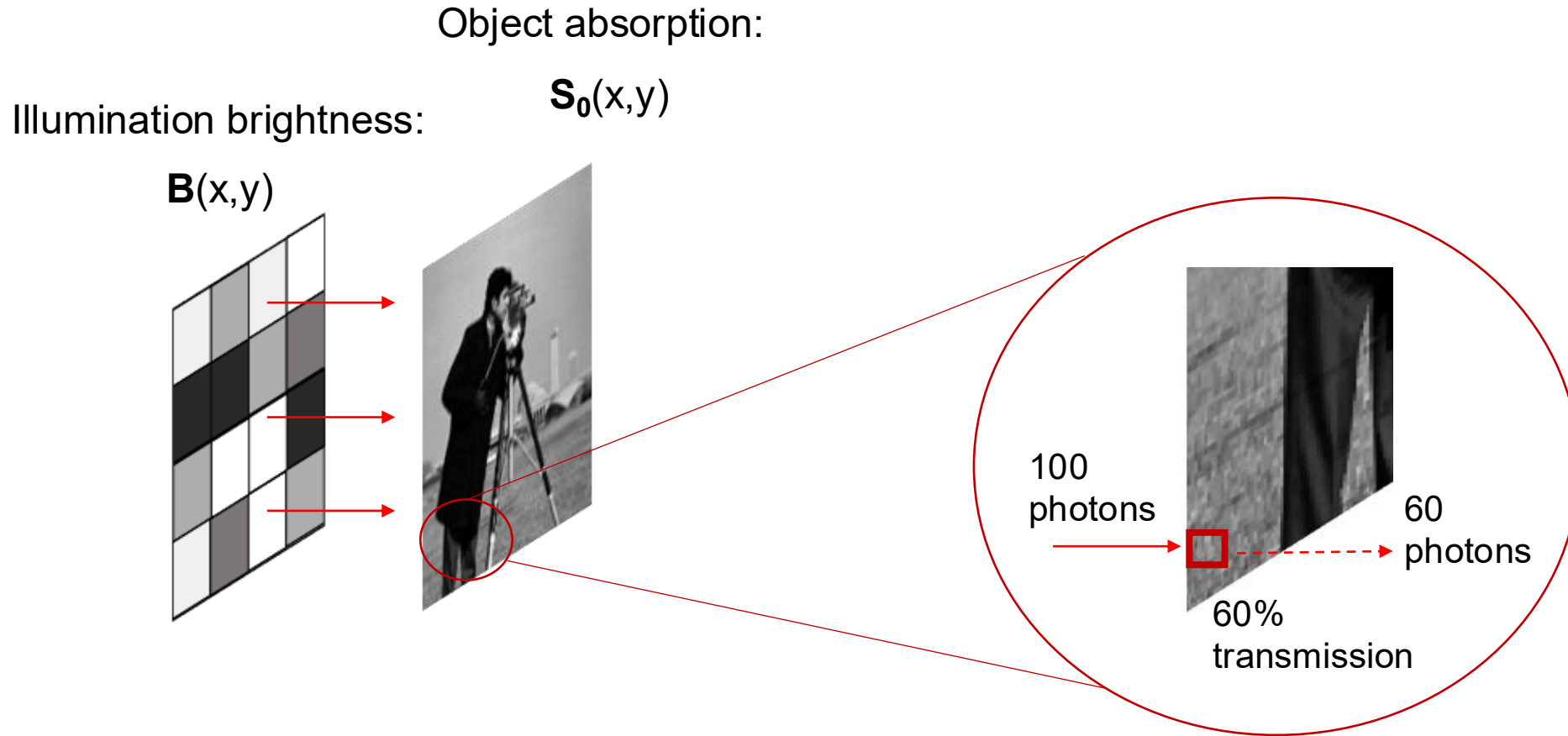
Illumination brightness:

$$B(x,y)$$



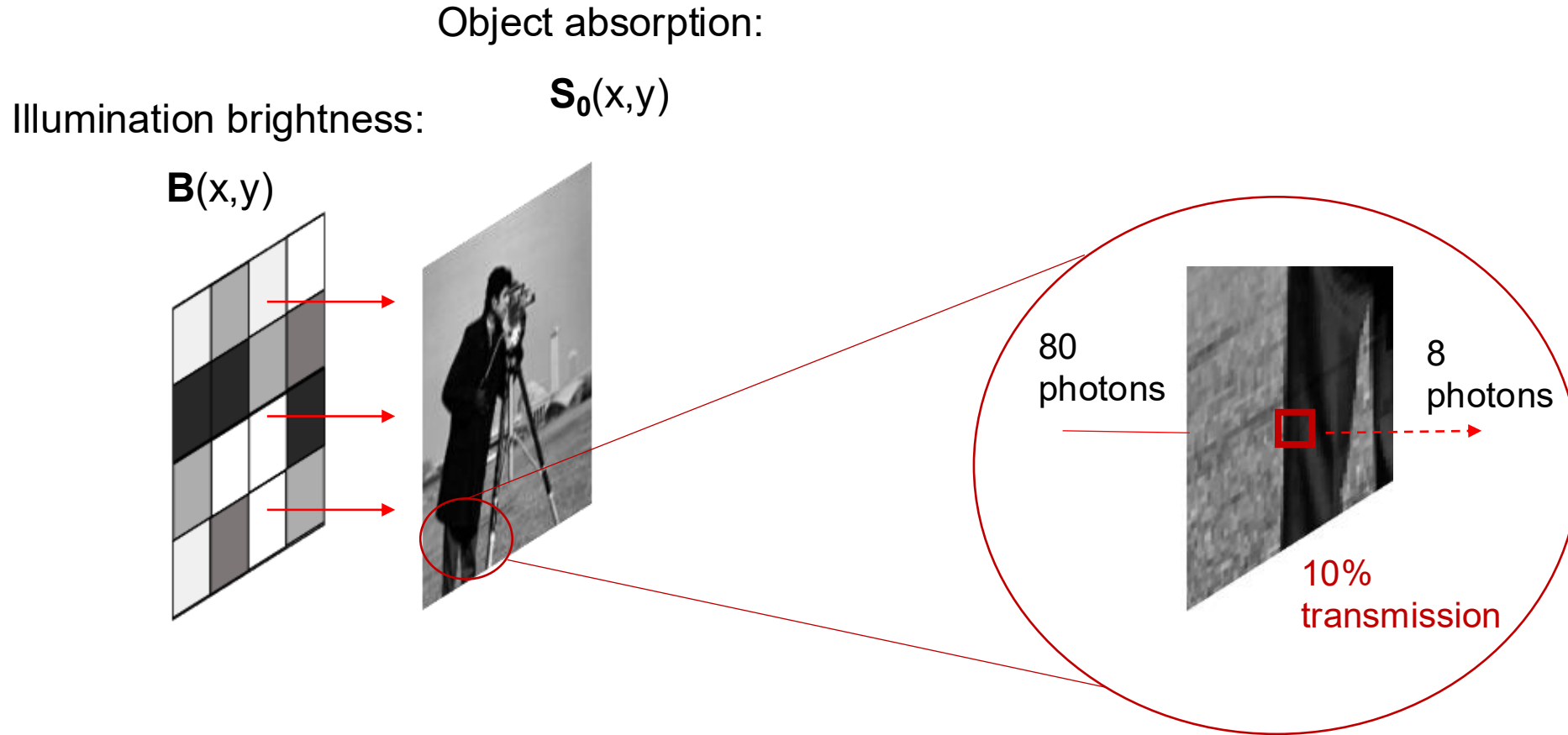
# Mathematical model of for incoherent image formation

- All quantities are real, and non-negative



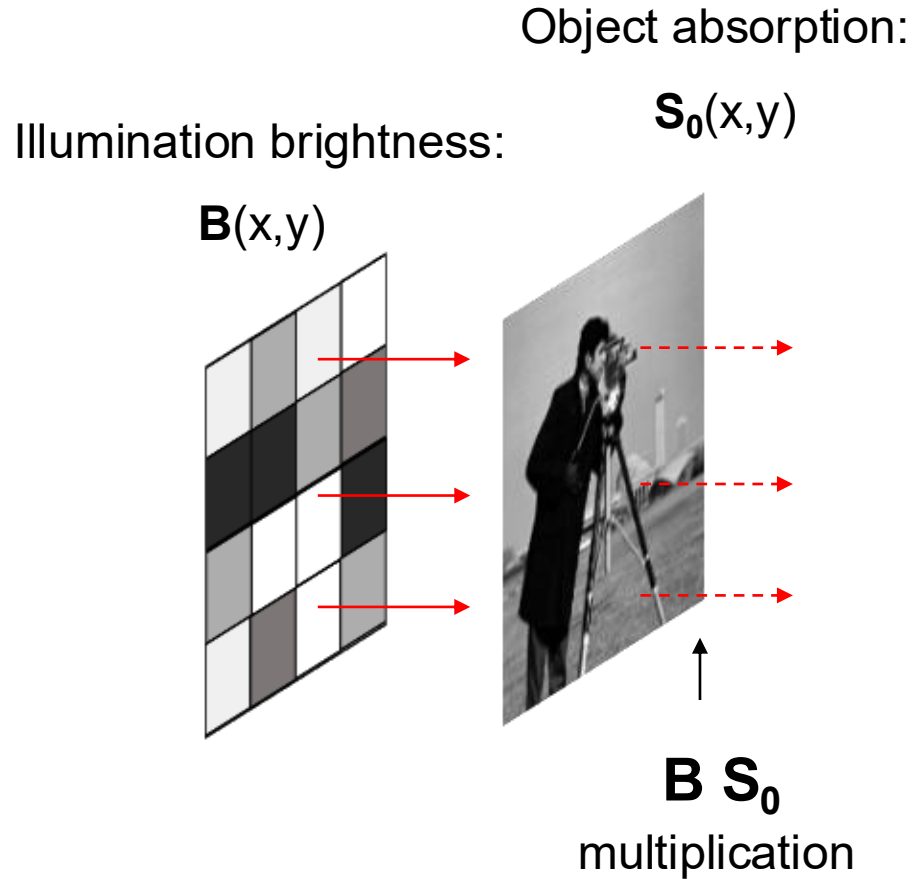
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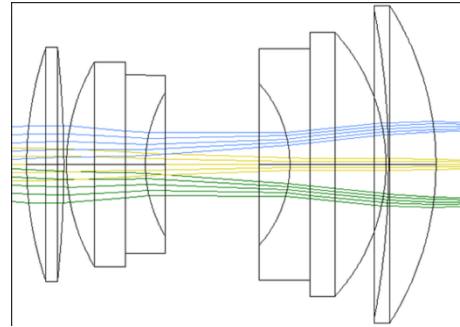
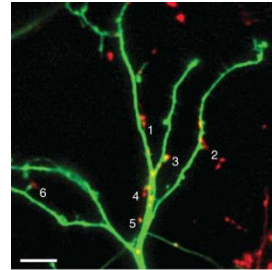
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# Summary of two models for image formation

- Interpretation #1: Radiation (*Incoherent*)
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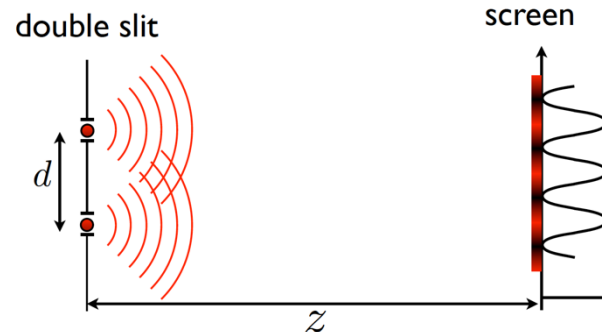
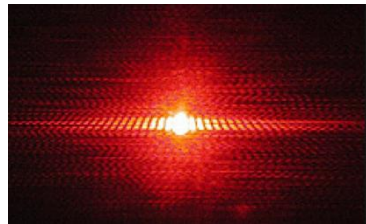


- Real, non-negative
- Models absorption and brightness

$$I_{\text{tot}} = I_1 + I_2$$

$$I_s = B S_0$$

- Interpretation #2: Electromagnetic wave (*Coherent*)
- Model: Waves



- Complex field
- Models Interference

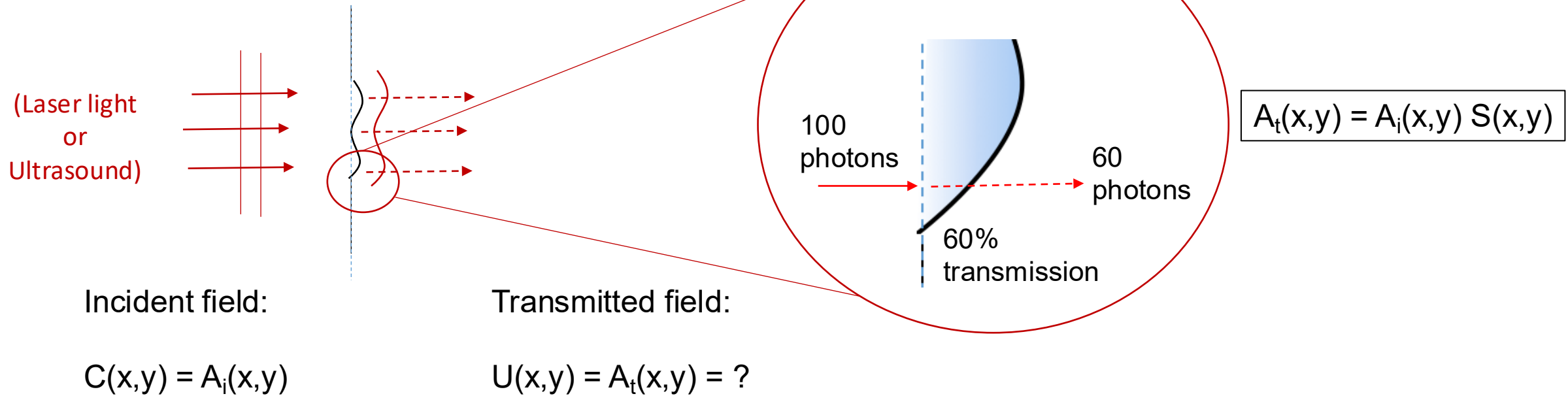
$$E_{\text{tot}} = E_1 + E_2$$

# Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Point #1: Amplitudes behave just like before

Sample absorption =  $S(x,y)$



# Mathematical model of for coherent image formation

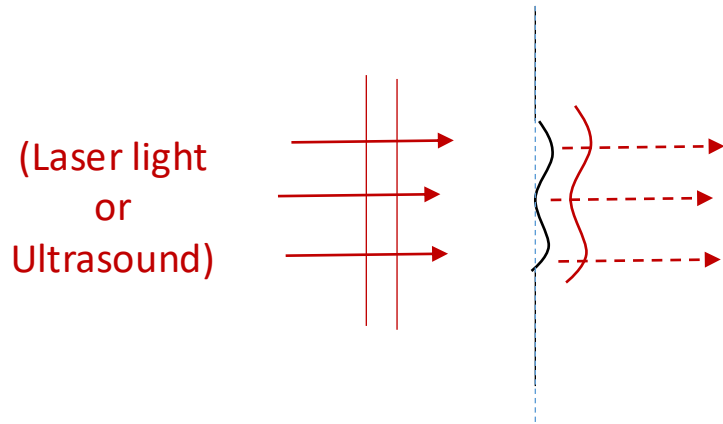
- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption =  $S(x,y)$

Sample phase delay =  $\exp[ik\phi(x,y)]$

New: complex phase delay

- Needed to represent wave
- Represents wave delay across space



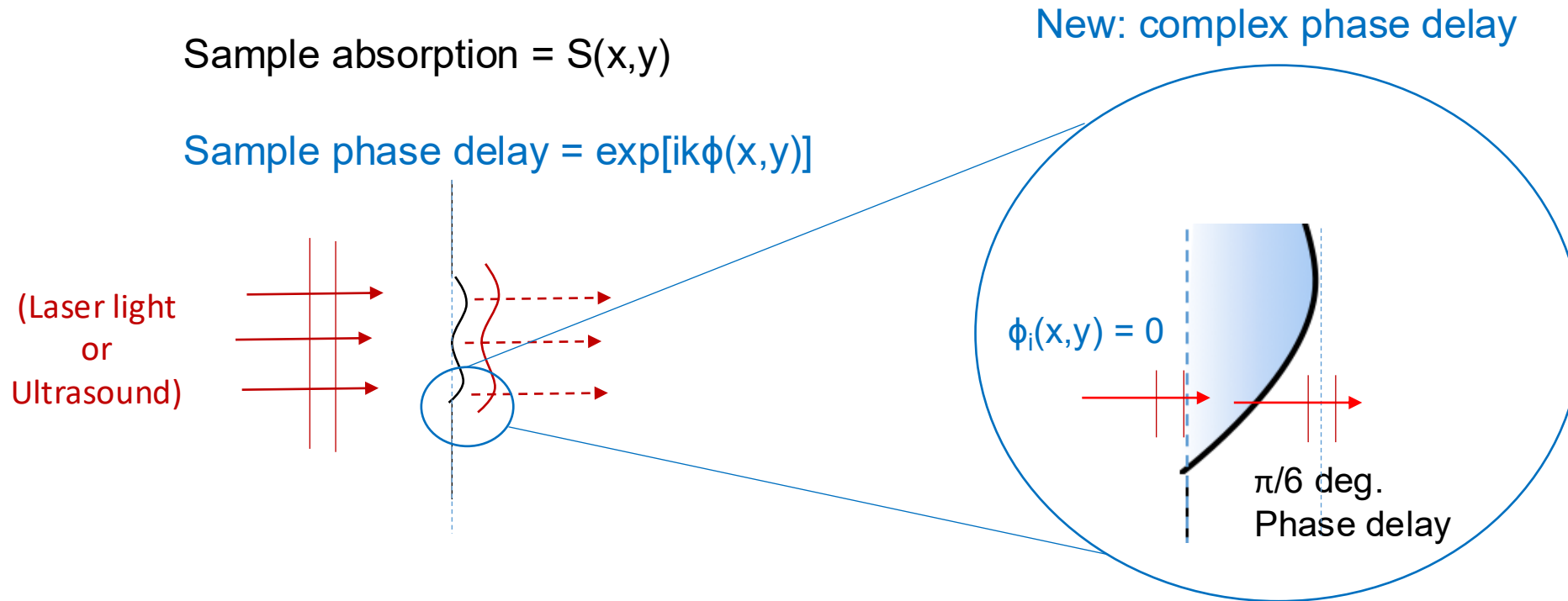
Incident field:

Transmitted field:

$$C(x,y) = A_i(x,y) \exp[ik\phi_i(x,y)] \quad U(x,y) = A_i(x,y) S(x,y) \exp[ik\phi_t(x,y)]$$

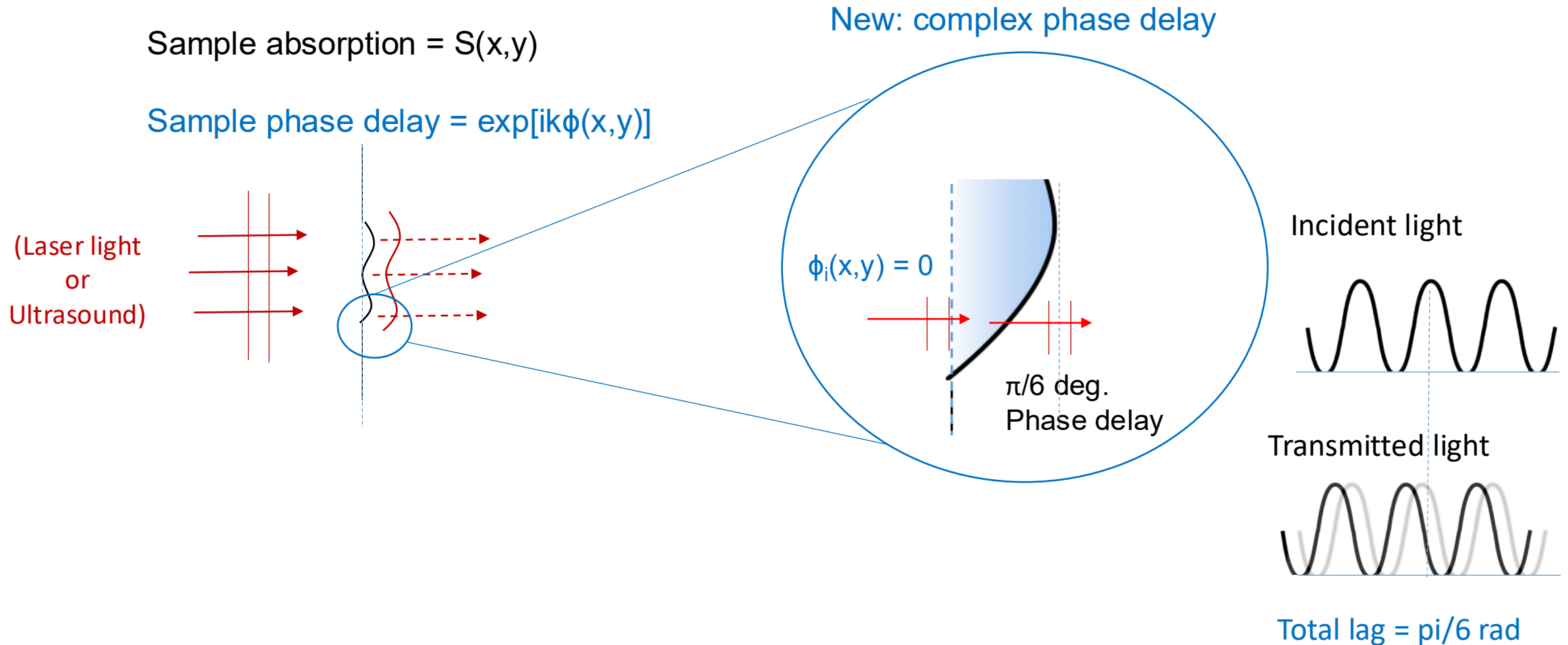
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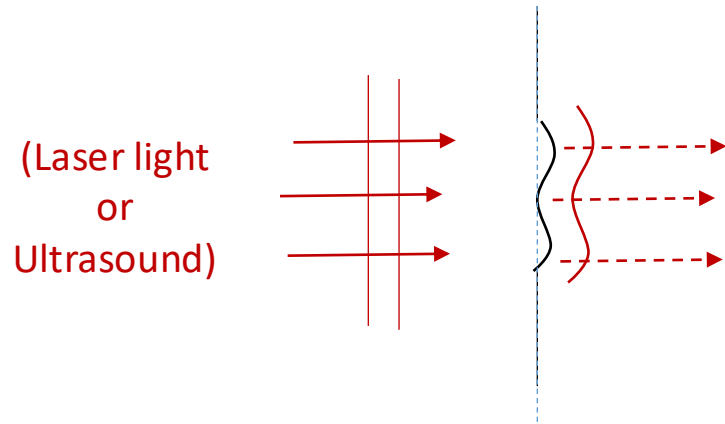


# Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption =  $S(x,y)$

Sample phase delay =  $\exp[ik\phi(x,y)]$



Output phase is sum of phase delays, product of phasors

$$\phi_t(x,y) = \phi(x,y) + \phi_i(x,y)$$

$$e^{ik\phi_t(x,y)} = e^{ik\phi_i(x,y)} \times e^{ik\phi(x,y)}$$

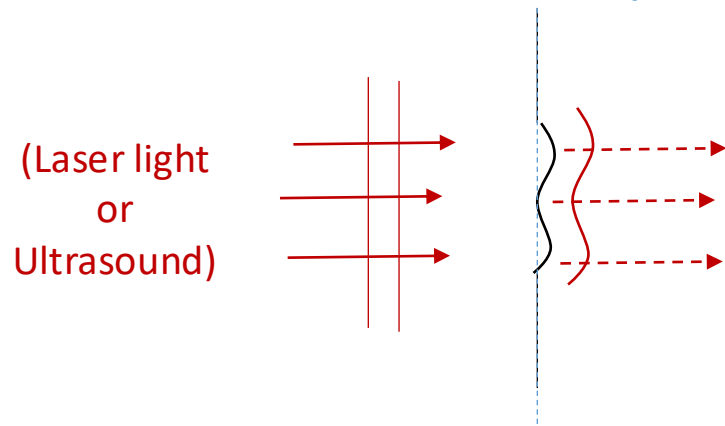
Multiply phases!

# Mathematical model of for coherent image formation

- Pretty much the same thing, but now we have an amplitude and a complex phase

Sample absorption =  $S(x,y)$

Sample phase delay =  $\exp[ik\phi(x,y)]$



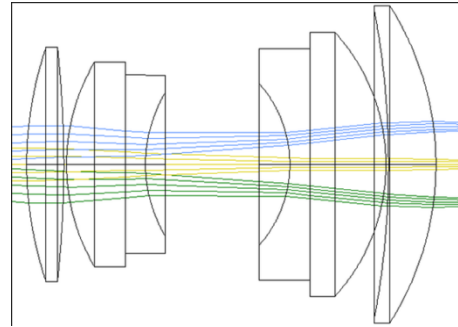
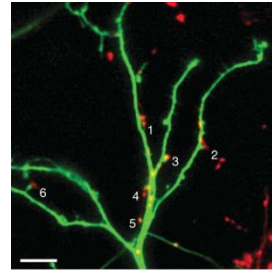
Conclusion:

Transmitted field = incident field x complex sample

$$A_t(x,y) \exp[ik\phi_t(x,y)] = A_i(x,y) \exp[ik\phi_i(x,y)] \times S(x,y) \exp[ik\phi(x,y)]$$

# Summary of two models for image formation

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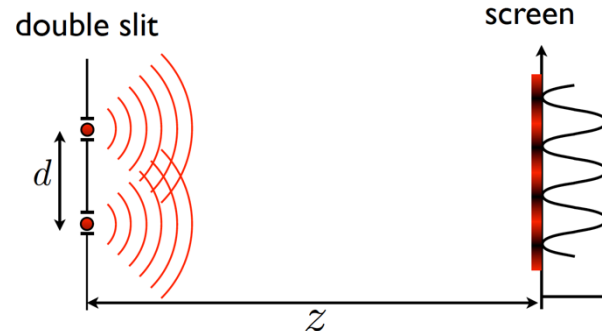
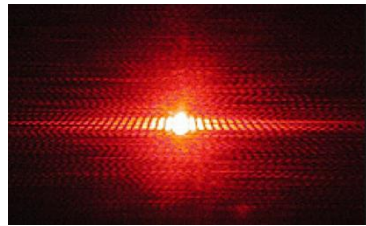


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$$I_{\text{tot}} = I_1 + I_2$$

$$I_s = B S_0$$

- Interpretation #2: Electromagnetic wave (*Coherent*)
- Model: Waves



- Complex field
- Models Interference

$$E_{\text{tot}} = E_1 + E_2$$

$$U = C S_0$$

$U, C$  and  $S$  are complex!

## **Additional Information about sample index of refraction, spatial frequency and Fourier optics**

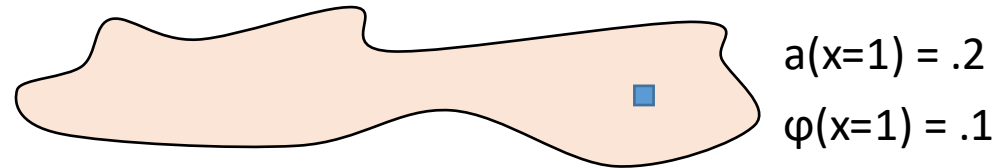
## Microscope illumination and sample index of refraction

So far: illuminate the sample and create a field that is equivalent to the sample's absorption and phase  
– why is this true???

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Sample index of refraction  $n(x,y,z) = 1 + ia(x) + \varphi(x)$

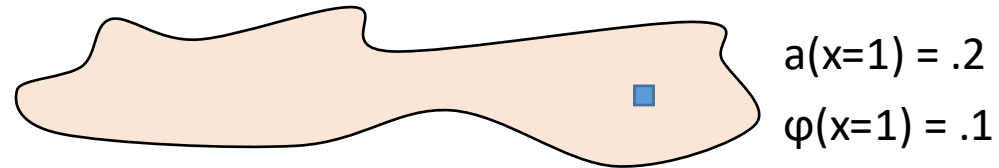


\*For more information, see D. Paganin, Coherent X-Ray Optics, Section 2.2

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**Thin sample approximation:**

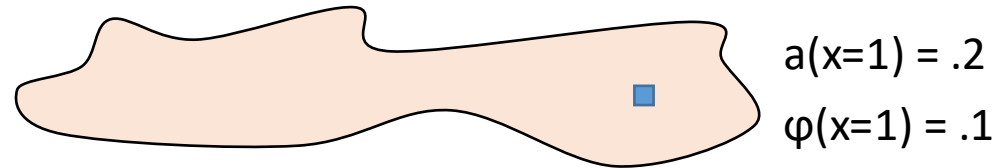
Sample's effect on light is multiplication with  $\exp[-ik * n(x,y)]$

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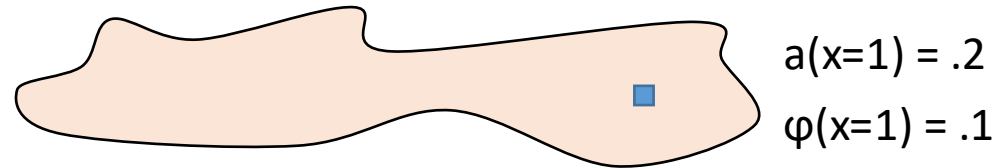
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In 1D: Emerging field  $U(x) =$  incident field  $U_i(x) * \text{sample function } s(x)$

# Microscope illumination and sample index of refraction

So far: illuminate the sample and create a field that is equivalent to the sample's absorption and phase  
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## Thin sample approximation:

Sample's effect on light is multiplication with  $\exp[-ik * n(x,y)]$

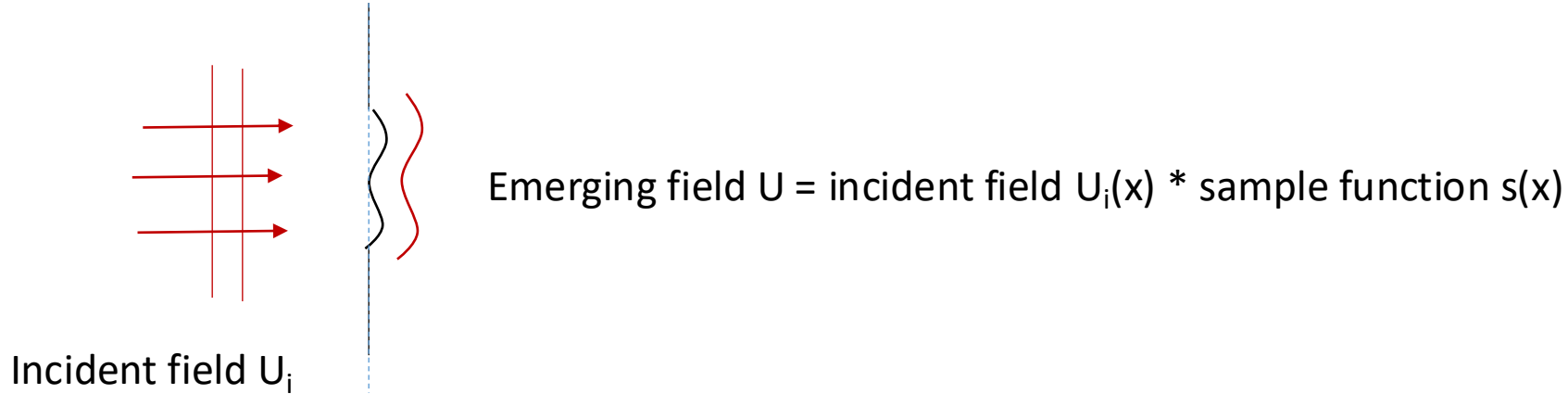
In 1D: Emerging field  $U(x) =$  incident field  $U_i(x) * \text{sample function } s(x) = \exp[-ik n(x)]$

$$U(x) = U_i(x) * \exp[-ik n(x)] = U_i(x) A(x) \exp[ik\varphi(x)] \quad A(x) = \exp[k a(x)]$$

↑
↑
absorption
phase shift: new term for laser

# Microscope illumination and sample index of refraction

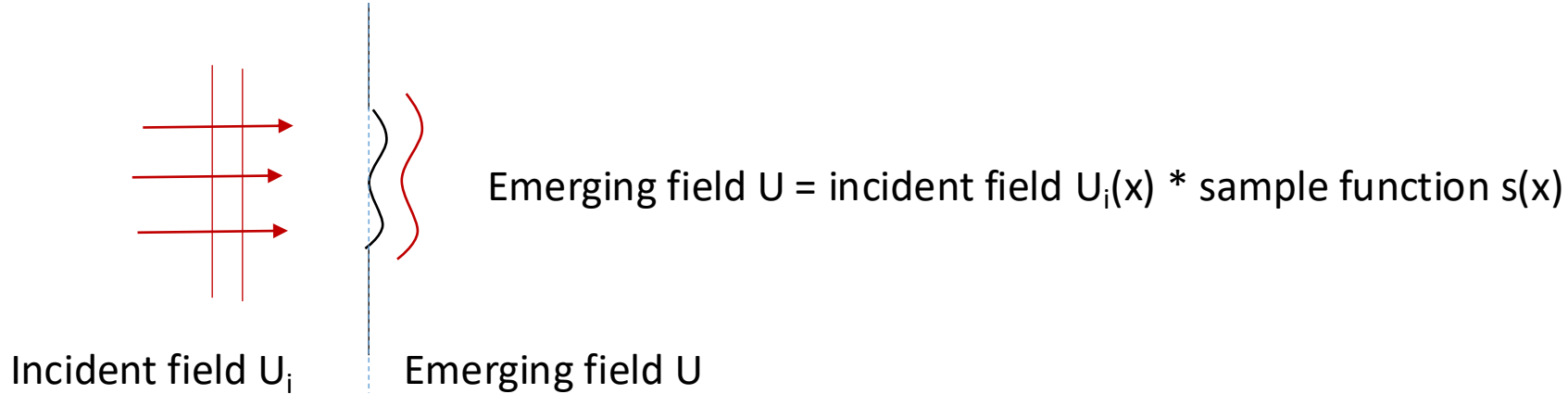
Sample absorption =  $A(x)$   
Sample phase =  $\exp[ik\phi(x)]$



## Microscope illumination and sample index of refraction

**Q:** When is the emerging field equal to the absorption and phase?

Sample absorption =  $A(x)$   
Sample phase =  $\exp[ik\phi(x)]$



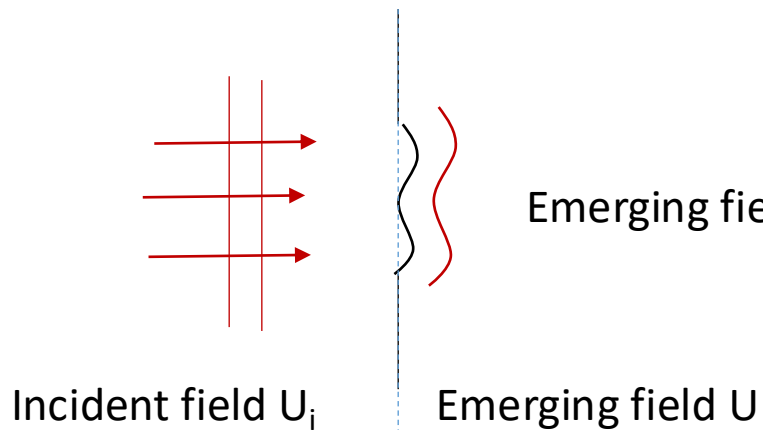
# Microscope illumination and sample index of refraction

**Q:** When is the emerging field equal to the absorption and phase?

Sample absorption =  $A(x)$   
 Sample phase =  $\exp[ik\phi(x)]$

**A:** When the incident wave = 1, means uniform in amplitude and phase:

$$U_i(x) = 1 \longrightarrow U(x) = A(x) \exp[ik\phi(x)]$$



Emerging field  $U =$  incident field  $U_i(x) * \text{sample function } s(x)$

# Microscope illumination and sample index of refraction

**Q:** When is the emerging field equal to the absorption and phase?

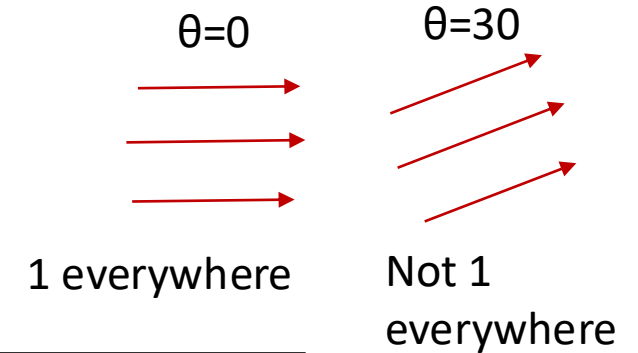
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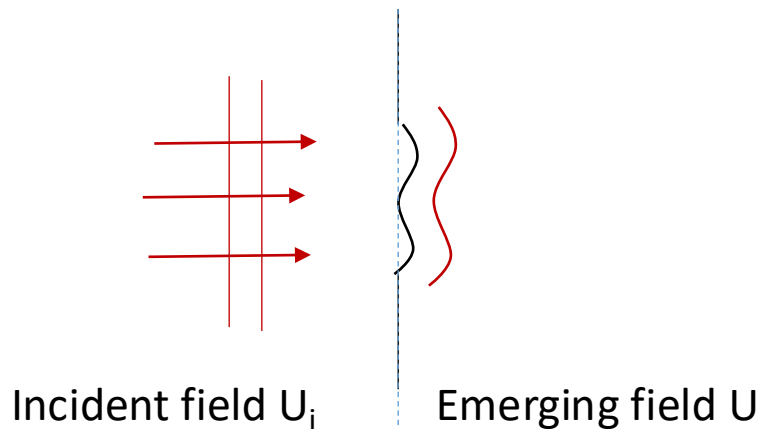
$$U_i(x) = 1 \longrightarrow U(x) = A(x) \exp[ik\phi(x)]$$

Plane wave  $U_i(x) = 1 * \exp(ik \cdot x)$

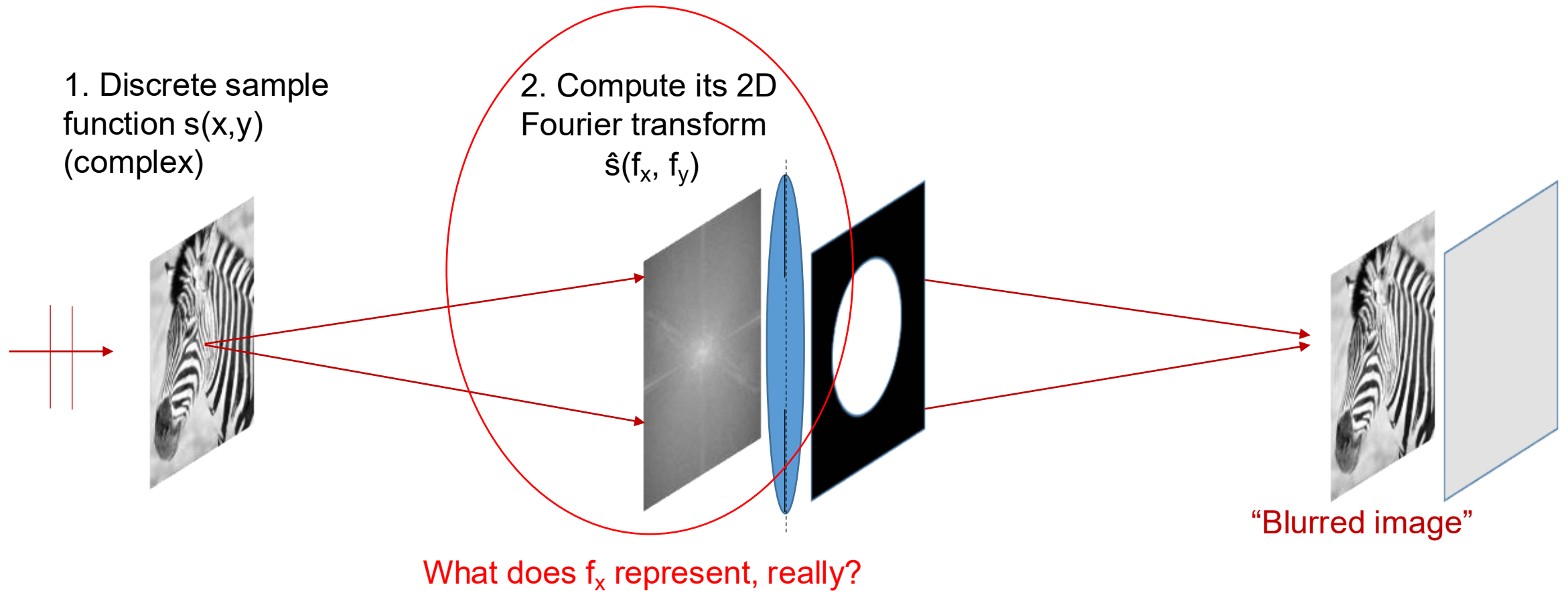
$$U_i(x) = \exp(ikx \sin(\theta))$$



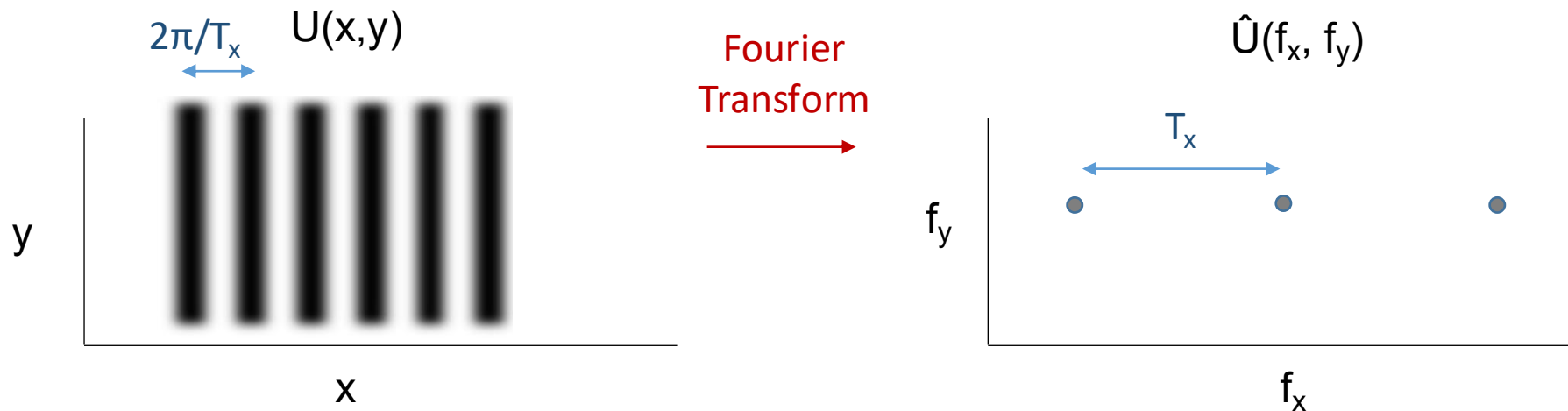
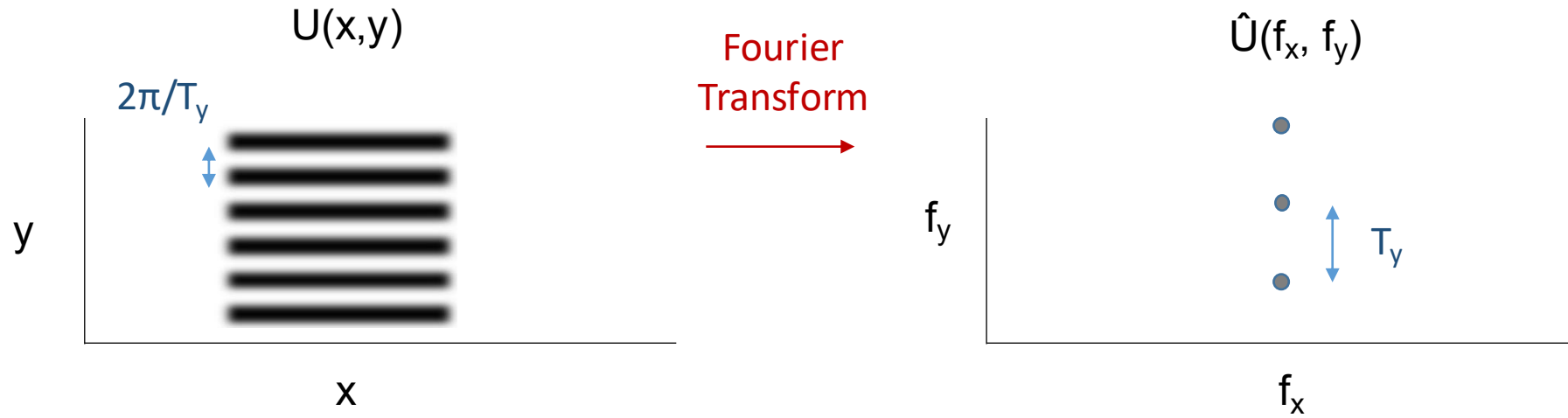
This is when incident wave hits the sample with  $\theta=0!$



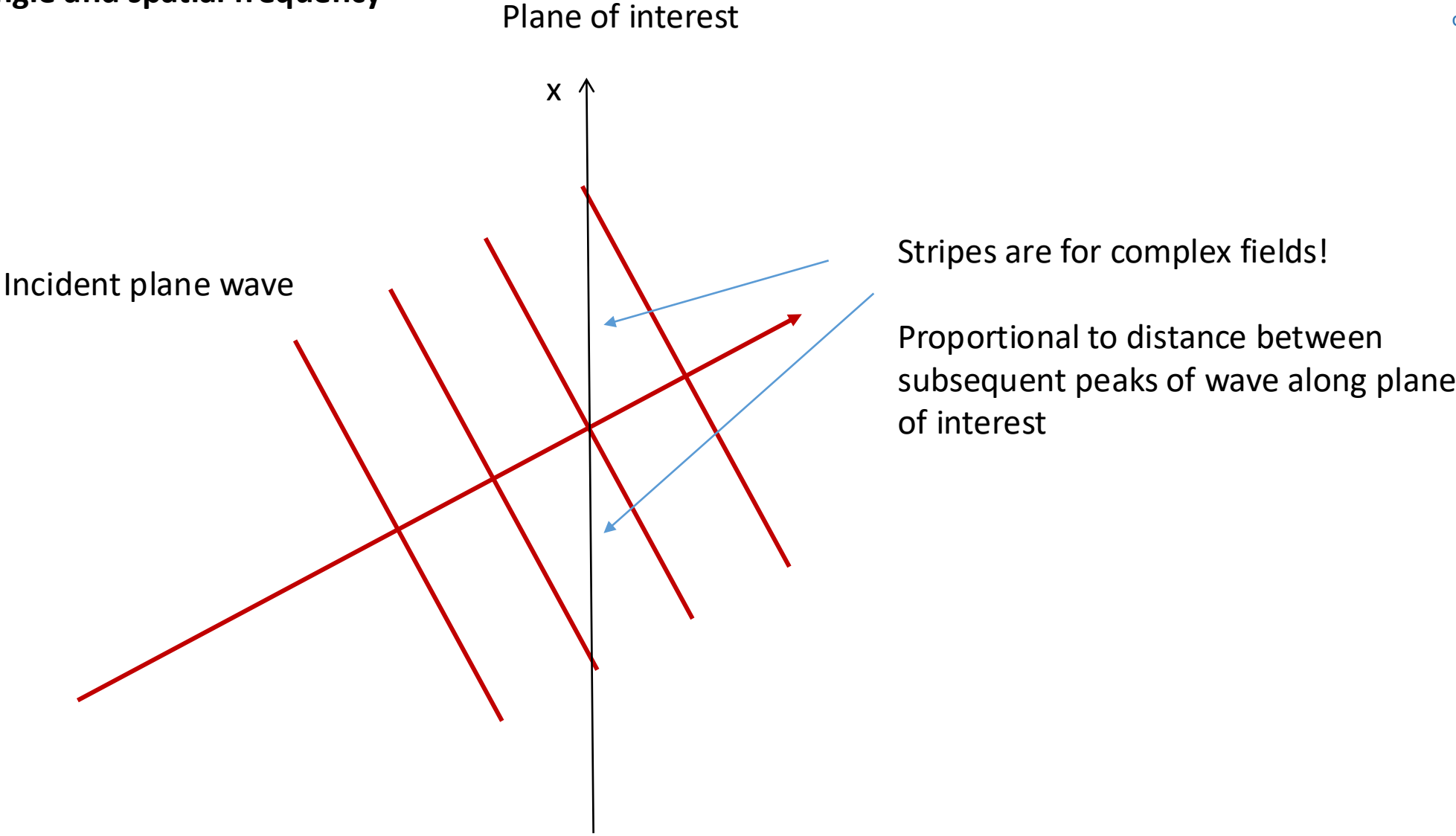
# Model of image formation for wave optics (coherent light):



# From before: Spatial frequencies = “stripes” within each image

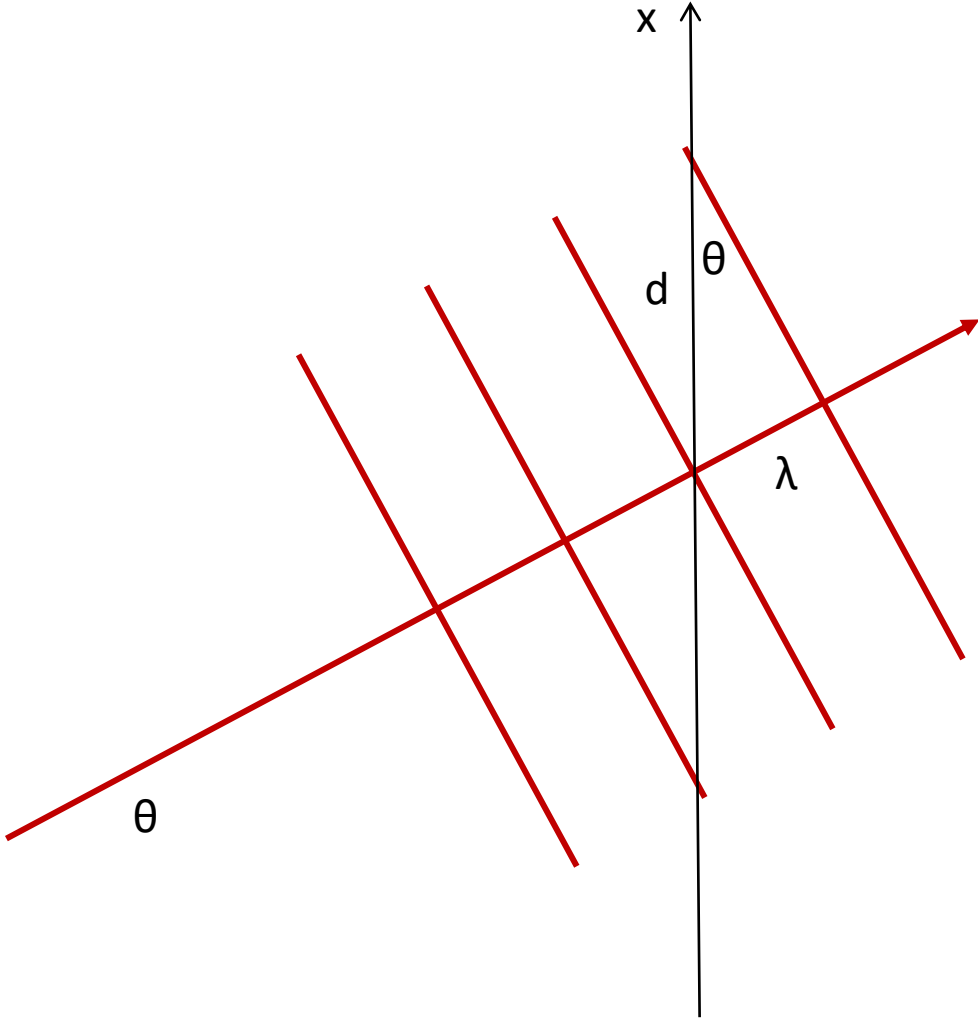


# Ray angle and spatial frequency



# Ray angle and spatial frequency

Plane of interest



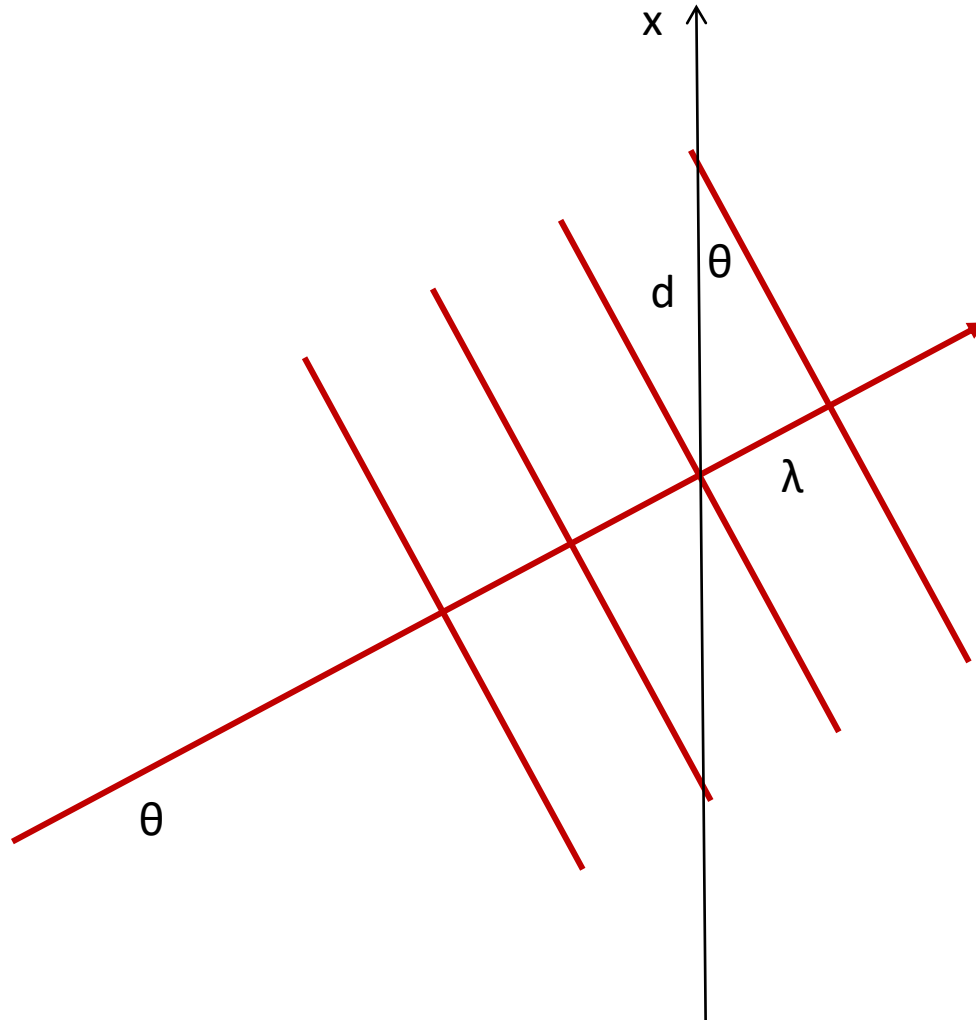
Distance to two crests = spatial period

$$\sin(\theta) = \lambda/d$$

$$d = \lambda / \sin(\theta)$$

# Ray angle and spatial frequency

Plane of interest



Distance to two crests = spatial period

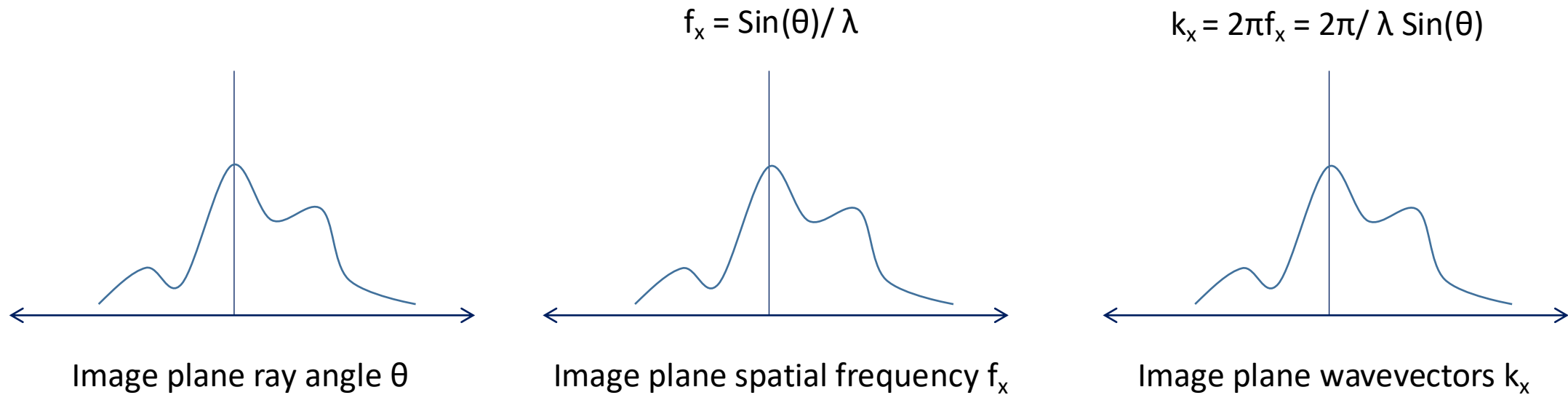
$$\sin(\theta) = \lambda/d$$

$$d = \lambda / \sin(\theta)$$

Spatial frequency = 1/spatial period  
(number of periods per unit length)

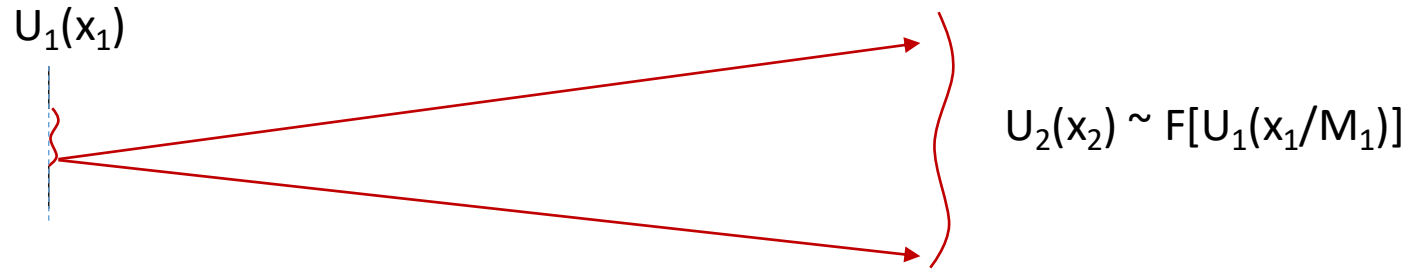
$$f_x = 1/d = \sin(\theta) / \lambda$$

## Equivalent coordinates in the Fourier domain and at the Fourier plane



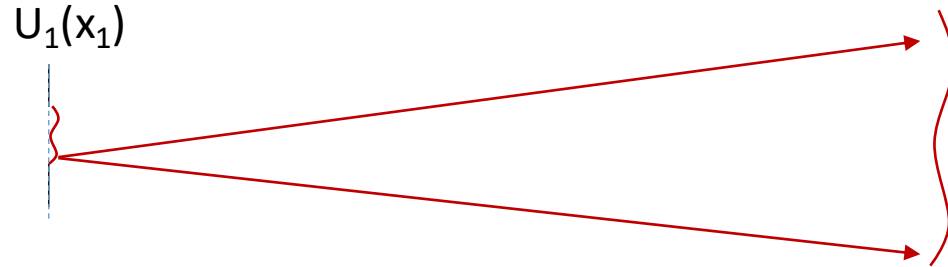
## General rules for applying the Fourier transform in optics

Situation 1: From an object to a plane “really far away”



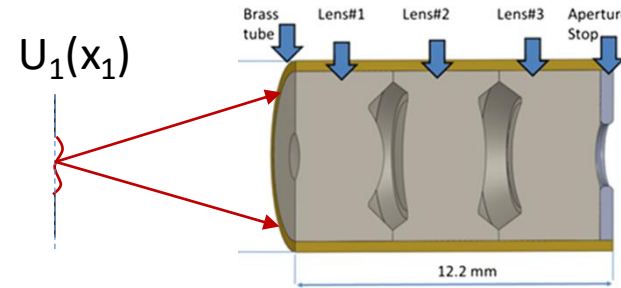
# General rules for applying the Fourier transform in optics

Situation 1: From an object to a plane “really far away”



$$U_2(x_2) \sim F[U_1(x_1/M_1)]$$

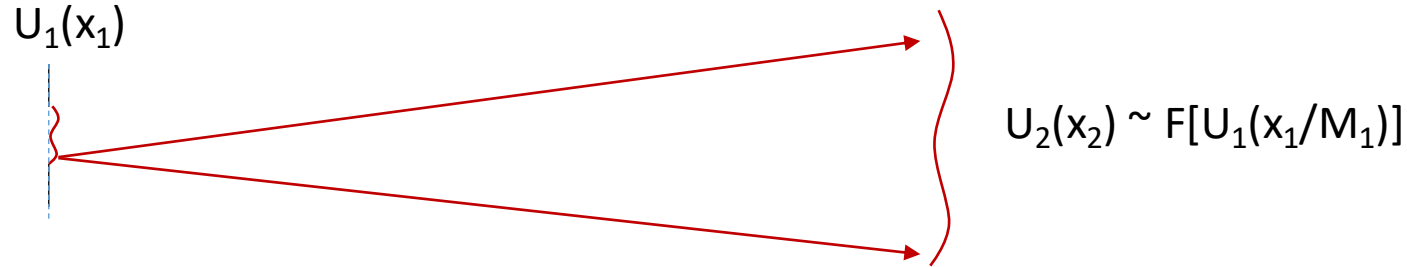
Situation 2: From an object to the back focal plane of the microscope objective lens



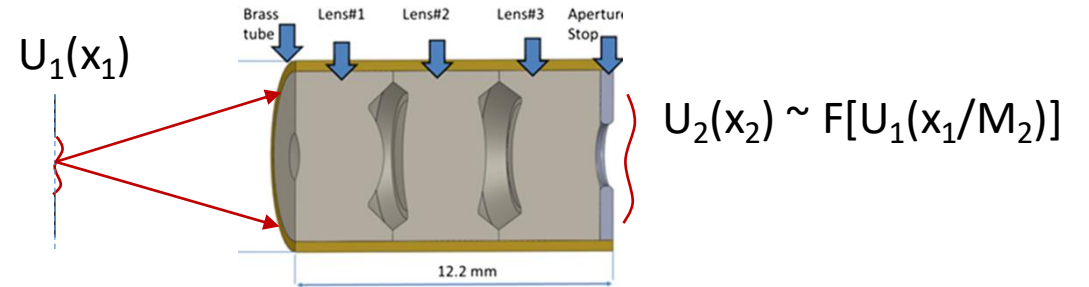
$$U_2(x_2) \sim F[U_1(x_1/M_2)]$$

# General rules for applying the Fourier transform in optics

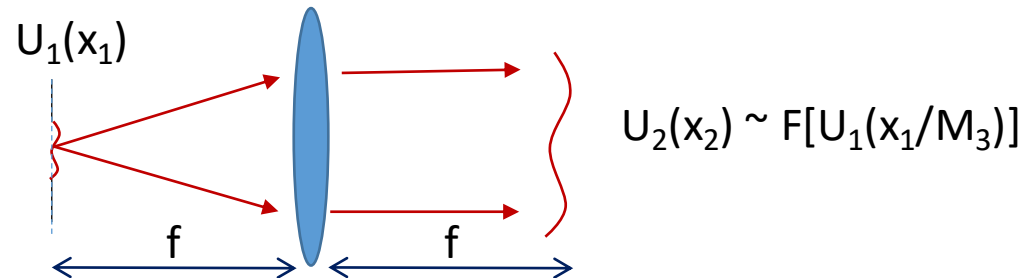
Situation 1: From an object to a plane “really far away”



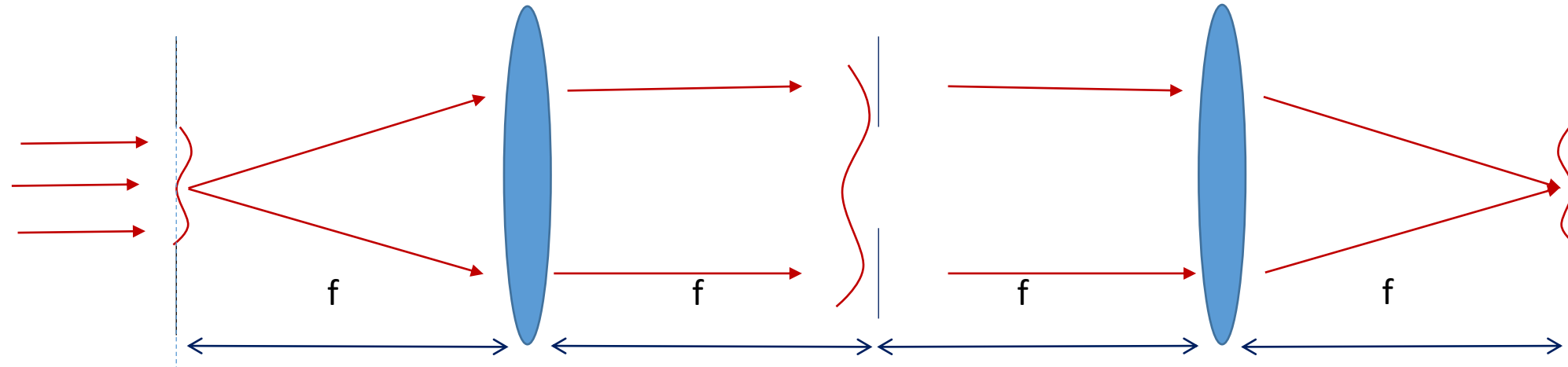
Situation 2: From an object to the back focal plane of the microscope objective lens



Situation 3: From an object to a plane 1 focal length away from a lens (1f-1f system)



## A more exact model: the 4f optical system



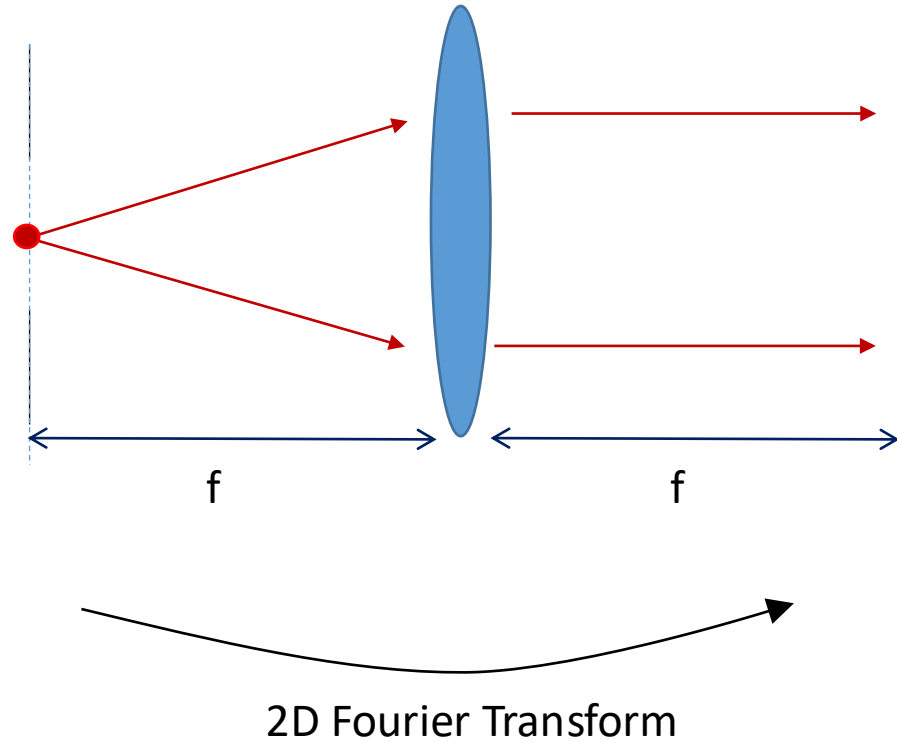
$$E_s(x_s, y_s, 0)$$

$$E'_a(x_d, y_d) = E_a(x_d, y_d)A(x_d, y_d)$$

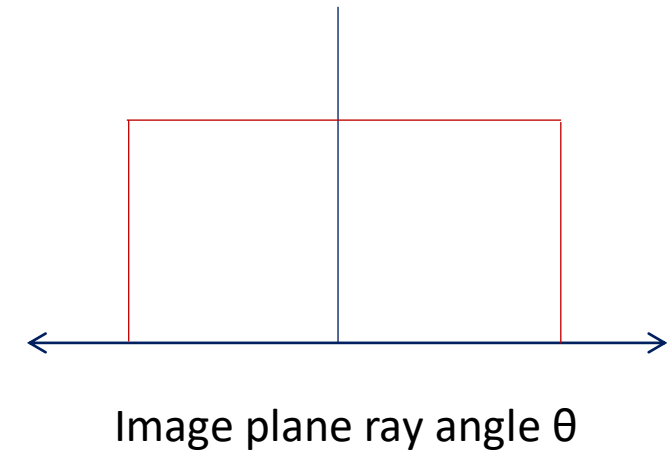
2D Fourier Transform

2D inverse Fourier Transform

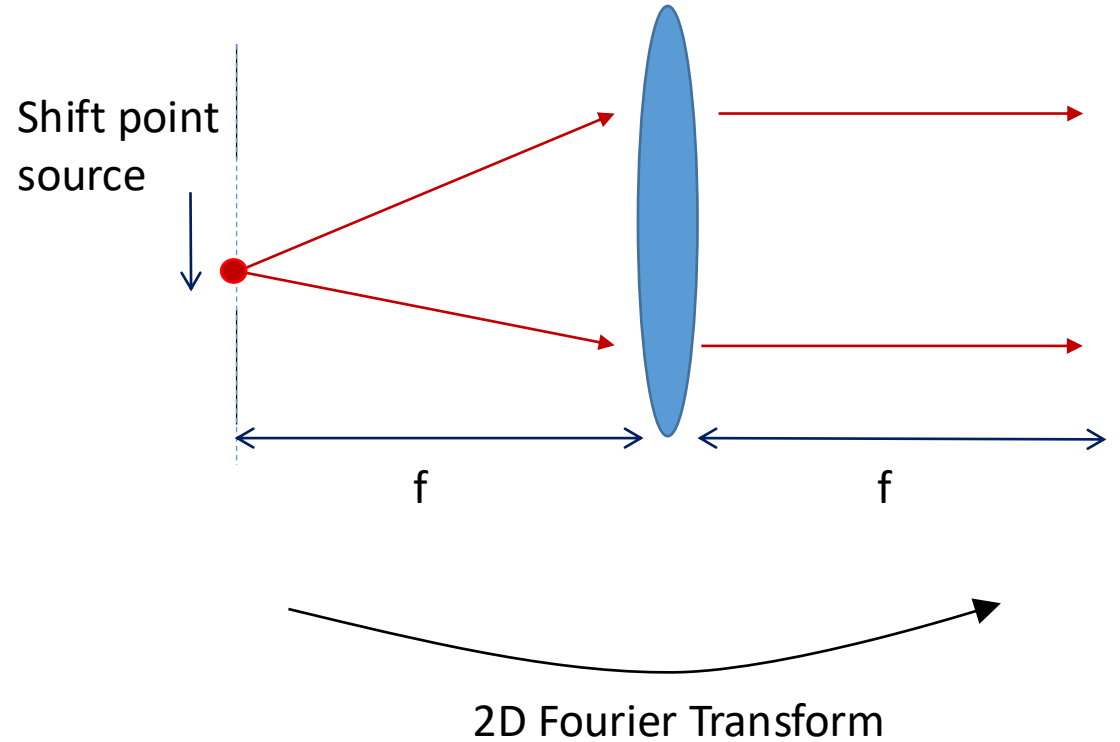
## A more exact model: the 4f optical system



The Fourier plane provides a measure of the **ray angles at the image plane**

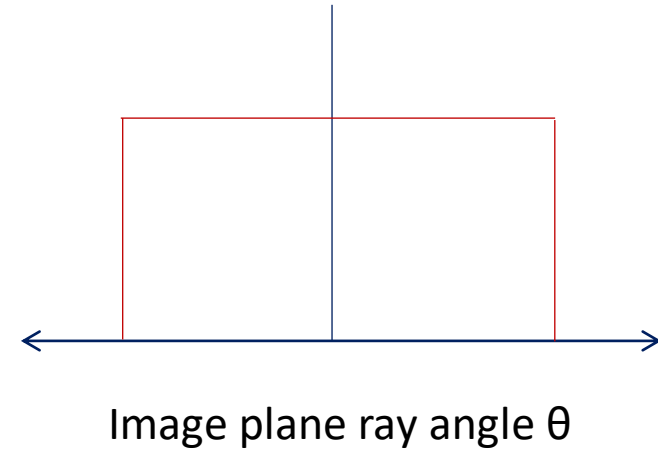


## A more exact model: the 4f optical system

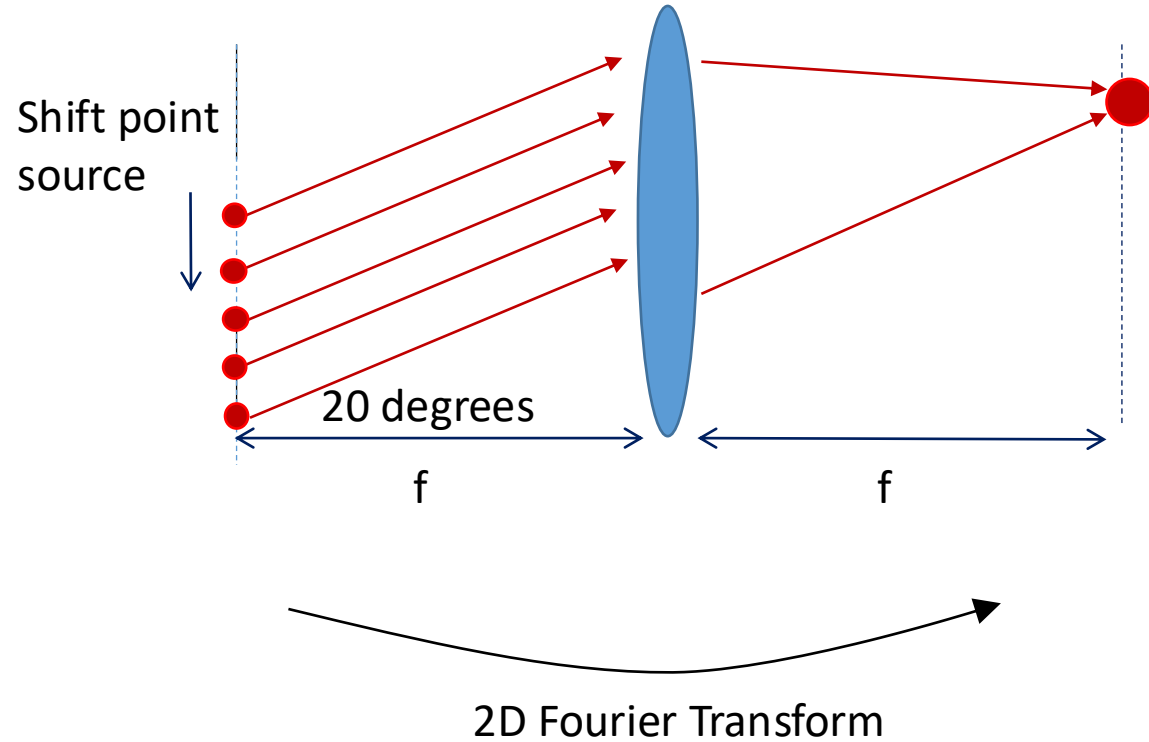


The Fourier plane provides a measure of the **ray angles at the image plane**

Doesn't contain info about spatial distribution light

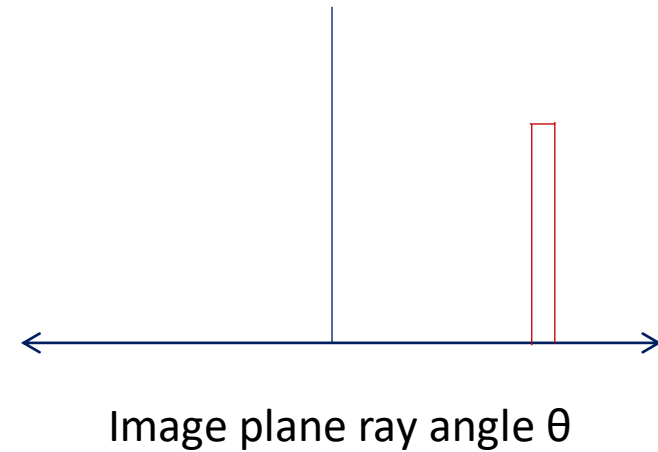


## A more exact model: the 4f optical system

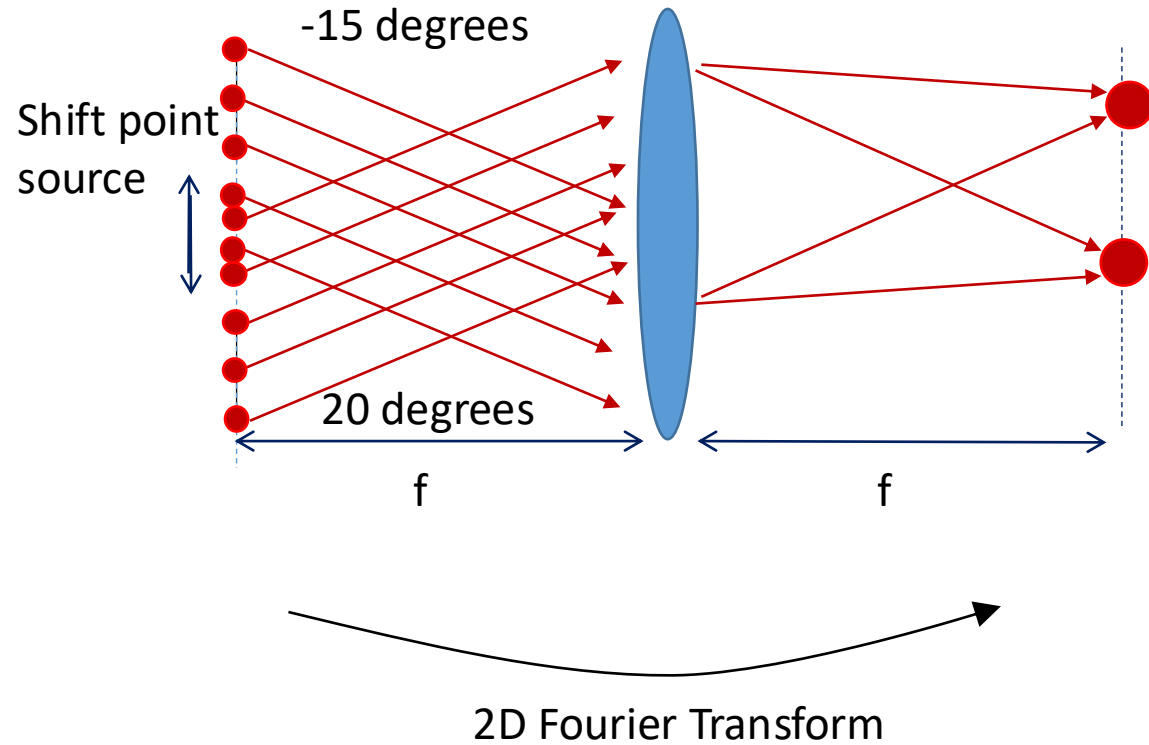


The Fourier plane provides a measure of the **ray angles at the image plane**

Rays are leaving image plane at +20 degrees

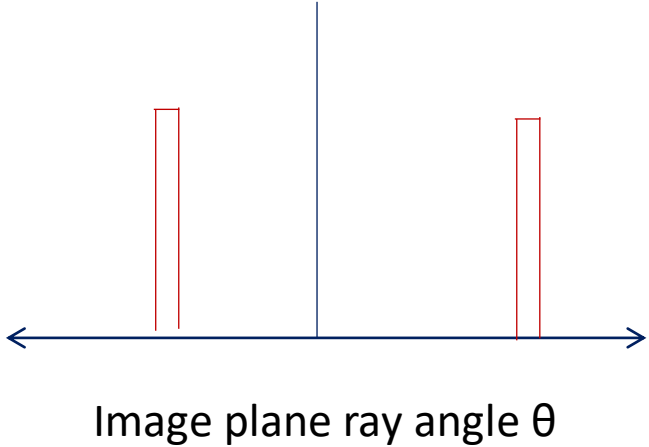


# A more exact model: the 4f optical system



The Fourier plane provides a measure of the **ray angles at the image plane**

Rays are coming in at +20 degrees and -15 degrees



You typically go between 4 functions to describe one imaging system:

