BME 548L: Homework 1

January 31, 2024

Due date: February 15, 2023 at 11:59pm

Please write up solutions to all problems (however you'd like) and submit them to Gradescope by the above due date. It is important that you show all of your steps - points will be deducted for simply writing the answer without showing any intermediate steps on how you arrived at your answer. You may work together to solve these problems, but please write up your own answer in your own way.

Remember that there is a coding component to this homework assignment as well, which can be found on the website: `https://deepimaging.github.io/homework/hw1`

Problem 1: Linear algebra refresher. We have two short-column vectors,

$$\mathbf{u} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ -7 \\ 1 \end{bmatrix}$$

For these two vectors, please compute,

(a) Their inner product, $\mathbf{u}^T\mathbf{v}$

(b) Their outer product (tensor product), $\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\mathbf{v}^T$

(c) Their Hadamard product (element-wise product), $\mathbf{u} \circ \mathbf{v}$

Next, consider the following two complex vectors of the form $Ae^{i\theta}$:

$$\mathbf{a} = \begin{bmatrix} 1 \\ e^{i\pi/2} \\ e^{-i\pi/3} \\ e^{-i\pi/3} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 1e^{i\pi} \\ 4e^{-i\pi/3} \\ 2e^{i\pi/2} \end{bmatrix}$$

For these two complex vectors, please compute,

(d) Their inner product, $\mathbf{a}^H\mathbf{b}$, where ^{H} denotes conjugate transpose

(e) The inner products, $\mathbf{a}^H\mathbf{a}$ and $\mathbf{b}^H\mathbf{b}$

(f) The Hadamard product (element-wise product) of $\mathbf{a} \circ \mathbf{a}^*$, where * denotes complex conjugate (no transpose) Finally, consider the following 3×3 grayscale image I:

$$\mathbf{I} = \begin{bmatrix} 3 & 9 & 2 \\ 4 & 7 & 8 \\ 5 & 6 & 1 \end{bmatrix}$$

Write an expression that obtains the average value of the first column of the image by using matrix multiplication operations, and carry out these operations for verification. Next, repeat this after vectorizing (or flattening) I using row-major ordering conventions.

Problem 2: Convolutions in 1D. Consider the following two short vectors (written out as row vectors to save space):

$$\mathbf{u} = [1, -1] \tag{1}$$

$$\mathbf{v} = [6, 5, 4, 3, 2, 1] \tag{2}$$

(a) Compute the discrete convolution, $\mathbf{w} = \mathbf{u} * \mathbf{v}$. You should first try to do this by hand, and then if you'd like, you can use a computer as an aid to check your result.

(b) The above convolution can be expressed as a matrix-vector product, $\mathbf{w} = \mathbf{U}\mathbf{v}$, where the matrix $\mathbf{U} \in \mathbb{R}^{7\times 6}$ holds the blur kernel $\mathbf{u}$. Write out $\mathbf{U}$. What generalized operation is it performing to the vector $\mathbf{v}$? Note: I am looking for something besides a convolution here - this particular convolution kernel is approximating a particular mathematical operation that we perform all the time. It is helpful to ignore the "edge" effects to establish what this particular operation of $\mathbf{U}$ on $\mathbf{v}$ is.

(c) One can also try to "undo" a convolution, which is called a deconvolution. This is helpful, for example, to take blurry images (that have been already convolved with a blur kernel) and to try to remove the effects of the blur, which can help create clearer images. Determine the deconvolution matrix $\mathbf{D}$ that undoes the operation of $\mathbf{U}$ to recover $\mathbf{v}$ from $\mathbf{w}$. What generalized operation is $\mathbf{D}$ performing?

(d) (bonus problem) Is $\mathbf{D}\mathbf{D}^T$ invertible? Why? What does its inverse represent?

Problem 3: Fourier transforms. Prove this important Fourier transform theorem:

$$\mathcal{F}\left[U(x) * V(x)\right] = \mathcal{F}\left[U(x)\right] \mathcal{F}\left[V(x)\right] \tag{3}$$

where $*$ denotes convolution.

Problem 4: Quick and conceptual. For the following problems, please provide a sketch and short explanation of the result. We are looking mostly for a correct conceptual understanding of these questions.

(a) Let's consider an optical field $U(x, y)$ that has passed through a camera lens and has reached the plane right before an optical detector. As we'll learn, the camera lens will block a lot of the high spatial frequencies within $U(x, y)$. Let's assume that at this plane, the maximum spatial frequency contained within $U(x, y)$ is 0.125 μm^{-1}. Furthermore, let's assume that the image sensor that detects the incoming optical field acts like a comb function, as we discussed in class, and has a pixel pitch of 10 μm in both x and y (this means that the interval spacing of the comb sampling is 10 μm). Would would one be able to faithfully re-create the original field $U(x, y)$ from the detected field (i.e., discretized field)? Why or why not? How about if the pixel pitch is a) 6 μm? b) 4 μm?

(b) Sketch the function $f(x) = x\cos(x)$ on the interval $x \in [-7, 7]$. Please mark the local minima and global minimum of $f(x)$ on this interval. If we were to perform gradient descent to determine the global minimum of this function, are we always guaranteed to locate it? What would be a good value of x to use as an initial estimate to perform gradient descent to find the global minimum? And what would be a bad initial value of x?

(c) Finally, let's consider the linear equation $f(x) = \text{sign}\left(\mathbf{w}^T\mathbf{x}\right)$, where the column vector $\mathbf{w} = [1, -3, 1]^T$, and our two-dimensional variable $\mathbf{x} = [1, x_1, x_2]^T$. While $\mathbf{x}$ does have 3 coordinates, we'll say it is two-dimensional because the first coordinate is fixed. "sign" is an operator such that $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = 0$ if $x = 0$, and $\text{sign}(x) = -1$ if $x < 0$. Show that the regions of $f(\mathbf{x}) = +1$ and $f(\mathbf{x}) = -1$ are separated by a line, and draw a sketch of the function $f(x)$ in the coordinate space (x_1, x_2) (Note: you don't have to draw the exactly correct slope or anything, just a conceptual sketch will do).